DEPARTMENT OF MATHEMATICS

NAME OF THE SUBJECT : TRANSFORMS & PARTIAL DIFFERENTIAL EQUATION

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UNIT – V

Z – TRANSFORM & DIFFERENCE EQUATION

Z-TRANSFORMS AND DEFFERENCE EQUATION

CLASS NOTES

Z-Transform of some basic functions:	
1.	$Z\left[a^n\right] = \frac{z}{z-a}$; $Z[1] = \frac{z}{z-1}$; $Z\left[(-a)^n\right] = \frac{z}{z+a}$
2.	$Z[n] = \frac{z}{(z-1)^2}$
3.	$Z\left[\frac{1}{n}\right] = \log\left[\frac{z}{z-1}\right]$
4.	$Z\left[\frac{1}{n+1}\right] = z\log\left[\frac{z}{z-1}\right]$
5.	$Z\left[\frac{1}{n-1}\right] = \frac{1}{z}\log\left[\frac{z}{z-1}\right]$
6.	$Z\left[\frac{1}{n!}\right] = e^{\frac{1}{z}}$
7.	$Z[\cos n\theta] = \frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}$
8.	$Z[\sin n\theta] = \frac{z\sin\theta}{z^2-2z\cos\theta+1}$

Inverse Z-Transforms:

The inverse Z-transform of $Z[f(n)] = F(z)$ is defined as $f(n) = Z^{-1}[F(z)]$.

The inverse Z-Transform of some basic functions:	
1.	$Z^{-1}\left[\frac{z}{z-1}\right] = 1$; $Z^{-1}\left[\frac{z}{z+1}\right] = (-1)^n$
2.	$Z^{-1}\left[\frac{z}{z-a}\right] = a^n$; $Z^{-1}\left[\frac{z}{z+a}\right] = (-a)^n$; $Z^{-1}\left[\frac{1}{z+a}\right] = a^{n-1}$
3.	$Z^{-1}\left[\frac{z^2}{(z-a)^2}\right] = (n+1)a^n$ For Eg. 1) $Z^{-1}\left[\frac{z}{(z-a)^2}\right] = (n-1+1)a^{n-1} = na^{n-1}$ 2) $Z^{-1}\left[\frac{1}{(z-a)^2}\right] = (n-2+1)a^{n-2} = (n-1)a^{n-2}$ 3) $Z^{-1}\left[\frac{z^2}{(z-1)^2}\right] = (n+1)1^n = n+1$ 4) $Z^{-1}\left[\frac{z}{(z-1)^2}\right] = (n-1+1)1^n = n$ 5) $Z^{-1}\left[\frac{1}{(z-1)^2}\right] = (n-2+1)1^n = n-1$
4.	$Z^{-1}\left[\frac{z^2}{z^2+a^2}\right] = a^n\cos\frac{n\pi}{2}$

5.	$Z^{-1}\left[\frac{z}{z^2+a^2}\right] = a^n\cos(n-1)\frac{\pi}{2} = a^n\cos\left[\frac{\pi}{2}-\frac{n\pi}{2}\right] = a^n\sin\frac{n\pi}{2}$

Finding Inverse Z-transform by method of **Partial Fractions**:

Rules of Partial Fractions:

1. Denominator containing Linear factors:

$$\frac{f(z)}{(z-a)(z-b)(z-c)\ldots} = \frac{A}{(z-a)} + \frac{B}{(z-b)} + \frac{C}{(z-c)} + \ldots$$

2. Denominator containing factors $(z-a)^n$:

$$\frac{f(z)}{(z-a)^n} = \frac{A}{(z-a)} + \frac{B}{(z-a)^2} + \frac{C}{(z-a)^3} + \ldots + \frac{D}{(z-a)^n}$$

3. Denominator contains a quadratic factor of the form az^2+bz+c (where a,b,c are constants):

$$\frac{f(z)}{az^2+bz+c} = \frac{A}{az^2+bz+c} + \frac{Bz}{az^2+bz+c}$$

(Or) $$\frac{f(z)}{az^2+bz+c} = \frac{Az+B}{az^2+bz+c}$$

1. **Find $Z^{-1}\left[\frac{z}{(z+1)(z-1)^2}\right]$ using the method partial fraction.**

Solution:

$$F(z) = \frac{z}{(z+1)(z-1)^2}$$

$$\frac{F(z)}{z} = \frac{1}{(z+1)(z-1)^2} \quad -----(1)$$

Now,

$$\frac{1}{(z+1)(z-1)^2} = \frac{A}{z+1} + \frac{B}{z-1} + \frac{C}{(z-1)^2}$$

$$1 = A(z-1)^2 + B(z+1)(z-1) + C(z+1)$$

Put $z=1 \Rightarrow 1 = 2C \Rightarrow \boxed{C = \frac{1}{2}}$

Put $z=-1, \Rightarrow 1 = 4A \Rightarrow \boxed{A = \frac{1}{4}}$

Put $z=0 \Rightarrow 1 = A - B + C \Rightarrow B = \frac{1}{4} + \frac{1}{2} - 1 \Rightarrow B = \frac{1+2-4}{4} \Rightarrow \boxed{B = \frac{-1}{4}}$

$$\frac{1}{(z+1)(z-1)^2} = \frac{\frac{1}{4}}{z+1} + \frac{\frac{-1}{4}}{z-1} + \frac{\frac{1}{2}}{(z-1)^2}$$

$$(1) \Rightarrow F(z) = \frac{1}{4}\frac{z}{z+1} - \frac{1}{4}\frac{z}{z-1} + \frac{1}{2}\frac{z}{(z-1)^2}$$

Taking Z^{-1} on both sides

$$(1) \Rightarrow Z^{-1}[F(z)] = \frac{1}{4}Z^{-1}\left[\frac{z}{z+1}\right] - \frac{1}{4}Z^{-1}\left[\frac{z}{z-1}\right] + \frac{1}{2}Z^{-1}\left[\frac{z}{(z-1)^2}\right]$$

$$\boxed{f(n) = \frac{1}{4}(-1)^n - \frac{1}{4}(1) + \frac{1}{2}n}$$

2. **Find** $Z^{-1}\left[\dfrac{z^{-2}}{\left(1+z^{-1}\right)^2\left(1-z^{-1}\right)}\right]$.

Solution:

$$F(z)=\frac{z^{-2}}{\left(1+z^{-1}\right)^2\left(1-z^{-1}\right)}=\frac{1}{z^2\frac{(z+1)^2}{z^2}\left(\frac{z-1}{z}\right)}$$

$$F(z)=\frac{z}{(z+1)^2(z-1)}$$

$$\frac{F(z)}{z}=\frac{1}{(z+1)^2(z-1)}\qquad -----(1)$$

$$\frac{1}{(z-1)(z+1)^2}=\frac{A}{z-1}+\frac{B}{z+1}+\frac{C}{(z+1)^2}$$

$$1=A(z+1)^2+B(z-1)(z+1)+C(z-1)$$

Put $z=1,\quad 1=4A \Rightarrow \boxed{A=\frac{1}{4}}$

Put $z=-1,\Rightarrow 1=-2c \Rightarrow \boxed{c=-\frac{1}{2}}$

Equating co-efficients of $z^2 \Rightarrow 0=A+B \Rightarrow \boxed{B=-\frac{1}{4}}$

$$(1)\Rightarrow \frac{F(z)}{z}=\frac{1}{4}\frac{1}{z-1}+\frac{-1}{4}\frac{1}{z+1}-\frac{1}{2}\frac{1}{(z+1)^2}$$

$$(1)\Rightarrow Z^{-1}[F(z)]=\frac{1}{4}Z^{-1}\left[\frac{z}{z-1}\right]-\frac{1}{4}Z^{-1}\left[\frac{z}{z+1}\right]-\frac{1}{2}Z^{-1}\left[\frac{z}{(z+1)^2}\right]$$

$$f(n)=\frac{1}{4}(1)^n-\frac{1}{4}(-1)^n+\frac{1}{2}n(-1)^n$$

$$\boxed{f(n)=\frac{1}{4}-\frac{1}{4}(-1)^n+\frac{1}{2}n(-1)^n}$$

3. **Find** $Z^{-1}\left[\dfrac{z^{-2}}{\left(1-z^{-1}\right)\left(1-2z^{-1}\right)\left(1-3z^{-1}\right)}\right]$.

Solution:

$$F(z)=\left[\frac{z^{-2}}{\left(1-z^{-1}\right)\left(1-2z^{-1}\right)\left(1-3z^{-1}\right)}\right]=\frac{\frac{1}{z^2}}{\left(1-\frac{1}{z}\right)\left(1-\frac{2}{z}\right)\left(1-\frac{3}{z}\right)}$$

$$=\frac{1}{z^2\left(\frac{z-1}{z}\right)\left(\frac{z-2}{z}\right)\left(\frac{z-3}{z}\right)}$$

$$F(z)=\frac{z}{(z-1)(z-2)(z-3)}$$

$$\frac{F(z)}{z}=\frac{1}{(z-1)(z-2)(z-3)}\qquad -----(1)$$

Now by Partial Fraction,

$$\frac{1}{(z-1)(z-2)(z-3)} = \frac{A}{z-1} + \frac{B}{z-2} + \frac{C}{z-3}$$

$$1 = A(z-2)(z-3) + B(z-1)(z-3) + C(z-1)(z-2)$$

Put $z = 2$, $\Rightarrow 1 = -B \Rightarrow \boxed{B = -1}$

Put $z = 1$, $\Rightarrow 1 = 2A \Rightarrow \boxed{A = \frac{1}{2}}$

Put $z = 3$, $\Rightarrow 1 = 2C \Rightarrow \boxed{C = \frac{1}{2}}$

$$(1) \Rightarrow F(z) = \frac{1}{2}\frac{z}{z-1} - \frac{z}{z-2} + \frac{1}{2}\frac{z}{z-3}$$

$$(1) \Rightarrow Z^{-1}[F(z)] = \frac{1}{2}Z^{-1}\left[\frac{z}{z-1}\right] - Z^{-1}\left[\frac{z}{z-2}\right] + \frac{1}{2}Z^{-1}\left[\frac{z}{z-3}\right]$$

$$f(n) = \frac{1}{2}(1)^n - (2)^n + \frac{1}{2}(3)^n$$

$$\boxed{f(n) = \frac{1}{2} - 2^n + \frac{1}{2}3^n}$$

4. **Find the Z-transform of $\frac{z^2+z}{(z-1)(z^2+1)}$ using partial fraction.**

Solution:

$$F(z) = \frac{z^2+z}{(z-1)(z^2+1)}$$

$$\frac{F(z)}{z} = \frac{z+1}{(z-1)(z^2+1)}$$

$$\frac{z+1}{(z-1)(z^2+1)} = \frac{A}{(z-1)} + \frac{B}{(z^2+1)} + \frac{Cz}{(z^2+1)}$$

$$z+1 = A(z^2+1) + B(z-1) + Cz(z-1)$$

Put $z = 1$, $\Rightarrow 2 = 2A \Rightarrow \boxed{A = 1}$

Equating co-efficients of $z^2 \Rightarrow 0 = A + C \Rightarrow \boxed{C = -1}$

Put $z = 0$, $\Rightarrow 1 = A - B \Rightarrow B = A - 1 = 1 - 1 = 0 \boxed{B = 0}$

$$\frac{F(z)}{z} = \frac{1}{(z-1)} + \frac{0}{(z^2+1)} + \frac{-z}{(z^2+1)}$$

$$F(z) = \frac{z}{(z-1)} - \frac{z^2}{(z^2+1)}$$

Put Z^{-1} on both sides

$$Z^{-1}[F(z)] = Z^{-1}\left[\frac{z}{z-1}\right] - Z^{-1}\left[\frac{z^2}{z^2+1}\right]$$

$$\boxed{f(n) = 1 - \cos\frac{n\pi}{2}} \qquad \because Z^{-1}\left[\frac{z^2}{z^2+a^2}\right] = \cos\frac{n\pi}{2}$$

Finding Inverse Z-transform by Residue Method:

By Inverse Z-Transforms $Z^{-1}[F(z)] = f(n)$

Procedure:

1. write $F(z)$ from given expression and write $F(z)z^{n-1}$

2. Find the poles by equating denominator to zero in $F(z)z^{n-1}$
3. Write the order of poles
4. Find the residue at these poles

Case i: If $z = a$ is pole of order 1 (or) simple pole then

$$\text{Res}\left[F(z)z^{n-1}\right]_{z=a} = \lim_{z\to a}(z-a)F(z)z^{n-1}$$

Case ii: If $z = a$ is pole of order m then $\text{Res}\left[F(z)z^{n-1}\right]_{z=a} = \dfrac{1}{|m-1} \lim_{z\to a} \dfrac{d^{m-1}}{dz^{m-1}}(z-a)^m F(z)z^{n-1}$

5. $f(n) =$ sum of residues of $F(z)z^{n-1}$

1 **Find $Z^{-1}\left[\dfrac{2z}{(z-2)(z^2+1)}\right]$ by the method of residues.**

Solution:

Let $F(z) = \dfrac{2z}{(z-1)(z^2+1)}$

$$F(z)z^{n-1} = \frac{2zz^{n-1}}{(z-1)(z^2+1)}$$

$$F(z)z^{n-1} = \frac{2z^n}{(z-1)(z+i)(z-i)} \quad ------(1)$$

Here $z = 1$, $z = i$ and $z = -i$ are poles of order 1.

1) $\text{Res}\left[F(z)z^{n-1}\right]_{z=a} = \lim_{z\to a}(z-a)F(z)z^{n-1}$

$$\text{Res}\left[F(z)z^{n-1}\right]_{z=1} = \lim_{z\to 1}(z-1)\frac{2z^n}{(z-1)(z+i)(z-i)}$$

$$= \lim_{z\to 1}\frac{2z^n}{(z+i)(z-i)}$$

$$= \frac{2(1)^n}{(1+i)(1-i)}$$

$$= \frac{2}{2} \qquad \because (1+i)(1-i) = 1^2 - i^2 = 1-(-1) = 1+1 = 2$$

$$\boxed{\text{Res}\left[F(z)z^{n-1}\right]_{z=1} = 1}$$

2) $\text{Res}\left[F(z)z^{n-1}\right]_{z=i} = \lim_{z\to a}(z-i)F(z)z^{n-1}$

$$\text{Res}\left[F(z)z^{n-1}\right]_{z=i} = \lim_{z\to i}(z-i)\frac{2z^n}{(z-1)(z-i)(z+i)}$$

$$= \lim_{z\to i}\frac{2z^n}{(z-1)(z+i)}$$

$$= \frac{2(i)^n}{(i-1)(i+i)}$$

$$= \frac{2(i)^n}{2i(i-1)}$$

$$= \frac{(i)^n}{i(i-1)} = \frac{(i)^n}{(i^2-i)} = \frac{(i)^n}{(-1-i)}$$

$$\boxed{\text{Res}\left[F(z)z^{n-1}\right]_{z=i} = \frac{-(i)^n}{(1+i)}}$$

	3) $\text{Res}\left[F(z)z^{n-1}\right]_{z=-i} = \lim_{z\to -i}(z+i)F(z)z^{n-1}$ $\text{Res}\left[F(z)z^{n-1}\right]_{z=-i} = \lim_{z\to -i}(z+i)\dfrac{2z^n}{(z-1)(z+i)(z-i)}$ $= \lim_{z\to -i}\dfrac{2z^n}{(z-1)(z-i)}$ $= \dfrac{2(-i)^n}{(-i-1)(-i-i)} = \dfrac{2(-i)^n}{(1+i)(2i)}$ $= \dfrac{(-i)^n}{(1+i)(i)} = \dfrac{(-i)^n}{(i+i^2)} = \dfrac{(-i)^n}{(i-1)}$ $\boxed{\text{Res}\left[F(z)z^{n-1}\right]_{z=-i} = \dfrac{(-i)^n}{(i-1)}}$ $f(n) = \text{sum of residues of } F(z)z^{n-1}$ $\boxed{f(n) = 1 - \dfrac{(i)^n}{(1+i)} + \dfrac{(-i)^n}{(i-1)}}$
2.	**Find the inverse Z-Transform of $\dfrac{z(z+1)}{(z-1)^3}$ by residue method.** **Solution:** Let $F(z) = \dfrac{z(z+1)}{(z-1)^3}$ $F(z)z^{n-1} = \dfrac{zz^{n-1}(z+1)}{(z-1)^3}$ $F(z)z^{n-1} = \dfrac{z^n(z+1)}{(z-1)^3}$ ------(1) $z=1$ is a pole of order 3 $\text{Res}\left[F(z)z^{n-1}\right] = \dfrac{1}{(m-1)!}\lim_{z\to a}\dfrac{d^{m-1}}{dz^{m-1}}(z-a)^m F(z)z^{n-1}$ $\text{Res}\left[F(z)z^{n-1}\right]_{z=1} = \dfrac{1}{(3-1)!}\lim_{z\to 1}\dfrac{d^2}{dz^2}(z-1)^3\dfrac{z^n(z+1)}{(z-1)^3}$ $= \dfrac{1}{2}\lim_{z\to 1}\dfrac{d^2}{dz^2}\left[z^{n+1}+z^n\right]$ $= \dfrac{1}{2}\lim_{z\to 1}\dfrac{d}{dz}\left[(n+1)z^n + nz^{n-1}\right]$ $= \dfrac{1}{2}\lim_{z\to 1}\left[(n+1)nz^{n-1} + n(n-1)z^{n-2}\right]$ $= \dfrac{1}{2}\lim_{z\to 1}\left[(n^2+n)(1)^{n-1} + (n^2-n)1^{n-2}\right]$ $= \dfrac{1}{2}\left[n^2+n+n^2-n\right]$ $\text{Res}\left[F(z)z^{n-1}\right]_{z=1} = \dfrac{1}{2}\left[2n^2\right]$ $\text{Res}\left[F(z)z^{n-1}\right]_{z=1} = n^2$

	$f(n) = \text{sum of residues of } F(z)z^{n-1} = n^2$
3.	**Find the inverse Z-transform of the function $\frac{z}{z^2+7z+10}$ by the method of residues.** **Solution:** $Z^{-1}\left[\frac{z}{z^2+7z+10}\right] = ?$ $F(z) = \frac{z}{z^2+7z+10} = \frac{z}{(z+2)(z+5)}$ $F(z)z^{n-1} = \frac{zz^{n-1}}{(z+2)(z+5)}$ $F(z)z^{n-1} = \frac{z^n}{(z+2)(z+5)}$ -------(1) Here $z=-2$ and $z=-5$ are pole of order 1 1) $\operatorname{Res}\left[F(z)z^{n-1}\right]_{z=a} = \lim_{z\to a}(z-a)F(z)z^{n-1}$ $\operatorname{Res}\left[F(z)z^{n-1}\right]_{z=-2} = \lim_{z\to -2}(z+2)\frac{z^n}{(z+2)(z+5)}$ $= \frac{(-2)^n}{(-2+5)} = \frac{(-2)^n}{3}$ $\boxed{\operatorname{Res}\left[F(z)z^{n-1}\right]_{z=-2} = \frac{(-2)^n}{3}}$ 2) $\operatorname{Res}\left[F(z)z^{n-1}\right]_{z=-5} = \lim_{z\to -5}(z+5)\frac{z^n}{(z+2)(z+5)}$ $= \frac{(-5)^n}{(-5+2)} = \frac{(-5)^n}{-3}$ $\boxed{\operatorname{Res}\left[F(z)z^{n-1}\right]_{z=-5} = \frac{-(-5)^n}{3}}$ $f(n) = \text{sum of residues of } F(z)z^{n-1}$ $\boxed{f(n) = \frac{(-2)^n}{3} - \frac{(-5)^n}{3} = \frac{1}{3}\left[(-2)^n - (-5)^n\right]}$
4.	**Find $Z^{-1}\left[\frac{z^{-2}}{(1+z^{-1})^2(1-z^{-1})}\right]$ by using residue method.** **Solution:** $F(z) = \frac{z^{-2}}{(1+z^{-1})^2(1-z^{-1})} = \frac{1}{z^2\left[\frac{z+1}{z}\right]^2\left[\frac{z-1}{z}\right]}$ $F(z) = \frac{z}{(z+1)^2(z-1)}$ $F(z)z^{n-1} = \frac{zz^{n-1}}{(z+1)^2(z-1)}$

	$F(z)z^{n-1} = \dfrac{z^n}{(z+1)^2(z-1)}$ - - - - - (1) Here $z = -1$ is pole of order 2, and $z = 1$ is pole of order 1 1) $\text{Res}\, F(z)z^{n-1} \big\|_{z=a} = \dfrac{1}{(m-1)!}\lim_{z\to a}\dfrac{d^{m-1}}{dz^{m-1}}(z-a)^m F(z)z^{n-1}$ $\text{Res}\, F(z)z^{n-1} \big\|_{z=-1} = \dfrac{1}{(2-1)!}\lim_{z\to -1}\dfrac{d^2}{dz^2}(z+1)^2\dfrac{z^n}{(z+1)^2(z-1)}$ $= \dfrac{1}{1!}\lim \dfrac{d}{dz}\left[\dfrac{z^n}{z-1}\right]$ $= \lim_{z\to -1}\left[\dfrac{(z-1)nz^{n-1} - z^n(1-0)}{(z-1)^2}\right]$ $= \dfrac{(-1-1)n(-1)^{n-1} - (-1)^n}{(-1-1)^2} = \dfrac{-2n(-1)^{n-1} - (-1)^n}{4} = \dfrac{(-1)^n}{4}[2n-1]$ $\text{Res}\, F(z)z^{n-1} \big\|_{z=-1} = \dfrac{(-1)^n}{4}[2n-1]$ 2) $\text{Res}\, F(z)z^{n-1} \big\|_{z=a} = \lim_{z\to a}(z-a)F(z)z^{n-1}$ $\text{Res}\, F(z)z^{n-1} \big\|_{z=1} = \lim_{z\to 1}(z-1)\dfrac{z^n}{(z+1)^2(z-1)}$ $= \lim_{z\to 1}\dfrac{z^n}{(z+1)^2} = \dfrac{1^n}{(1+1)^2} = \dfrac{1}{2}$ $\text{Res}\, F(z)z^{n-1} \big\|_{z=1} = \dfrac{1}{2}$ $f(n) = $ sum of residues of $F(z)z^{n-1}$ $\boxed{f(n) = \dfrac{(-1)^n}{4}[2n-1] + \dfrac{1}{2}}$
5.	**Using complex residue theorem evaluate** $Z^{-1}\left[\dfrac{9z^3}{(3z-1)^2(z-2)}\right]$ Solution: $Z^{-1}\left[\dfrac{9z^3}{(3z-1)^2(z-2)}\right] = Z^{-1}\left[\dfrac{9z^3}{9(z-\frac{1}{3})^2(z-2)}\right] = Z^{-1}\left[\dfrac{z^3}{(z-\frac{1}{3})^2(z-2)}\right]$ $F(z) = \dfrac{z^3}{(z-\frac{1}{3})(z-2)}$ $F(z)z^{n-1} = \dfrac{z^3 z^{n-1}}{(z-\frac{1}{3})^2(z-2)}$ $F(z)z^{n-1} = \dfrac{z^{n+2}}{(z-\frac{1}{3})^2(z-2)}$ Here $z = \dfrac{1}{3}$ are pole of order 2 and $z = 2$ is simple pole. 1) $\text{Res}\, F(z)z^{n-1} \big\|_{z=a} = \dfrac{1}{(m-1)!}\lim_{z\to a}\dfrac{d^{m-1}}{dz^{m-1}}(z-a)^m F(z)z^{n-1}$ here $m = 2$

$$\operatorname{Res}\left[F(z)z^{n-1}\right]_{z=\frac{1}{3}} = \lim_{z\to\frac{1}{3}} \frac{d}{dz}\left[(z-\tfrac{1}{3})^2 \frac{z^{n+2}}{(z-\frac{1}{3})^2 (z-2)}\right]$$

$$= \lim_{z\to\frac{1}{3}} \frac{d}{dz}\left[\frac{z^{n+2}}{z-2}\right]$$

$$= \lim_{z\to\frac{1}{3}} \left[\frac{(z-2)(n+2)z^{n+1} - z^{n+2}(1)}{(z-2)^2}\right]$$

$$= \lim_{z\to\frac{1}{3}} \left[\frac{z^{n+1}\left[(z-2)(n+2)-z\right]}{(z-2)^2}\right]$$

$$= \left[\frac{\left(\frac{1}{3}\right)^{n+1}\left[\left(\frac{1}{3}-2\right)(n+2)-\frac{1}{3}\right]}{\left(\frac{1}{3}-2\right)^2}\right]$$

$$\operatorname{Res}\left[F(z)z^{n-1}\right]_{z=\frac{1}{3}} = \frac{\left(\frac{1}{3}\right)^{n+1}\left[\frac{-5(n+2)}{3} - \frac{1}{3}\right]}{\left(\frac{-5}{3}\right)^2} = \frac{\left(\frac{1}{3}\right)^{n+1}\left[\frac{-5n-10-1}{3}\right]}{\frac{25}{9}}$$

$$= \frac{9}{25}\left(\frac{1}{3}\right)^n \left(\frac{1}{3}\right)\left[\frac{-5n-11}{3}\right] = \frac{-1}{25}\left(\frac{1}{3}\right)^n (5n+11)$$

$$\boxed{\operatorname{Res}\left[F(z)z^{n-1}\right]_{z=\frac{1}{3}} = \frac{-1}{25}\left(\frac{1}{3}\right)^n (5n+11)}$$

2) $\operatorname{Res}\left[F(z)z^{n-1}\right]_{z=2} = \lim_{z\to 2} (z-2)\left[\frac{z^{n+2}}{(z-2)(z-\frac{1}{3})^2}\right]$

$$\operatorname{Res}\left[F(z)z^{n-1}\right]_{z=2} = \frac{2^{n+2}}{(2-\frac{1}{3})^2} = \frac{9}{25}2^{n+2}$$

$$\boxed{\operatorname{Res}\left[F(z)z^{n-1}\right]_{z=2} = \frac{9}{25}2^{n+2}}$$

$f(n)$ = sum of residues of $F(z)z^{n-1}$

$$\boxed{f(n) = f(n) = \frac{9}{25}2^{n+2} + \frac{-1}{25}\left(\frac{1}{3}\right)^n (5n+11)}$$

Finding Inverse Z-transform by Convolution theorem:

Convolution of two sequences:

If $\{f(n)\}$ and $\{g(n)\}$ are any two sequences then its convolution is defined by

$$f(n) * g(n) = \sum_{k=0}^{n} f(k)g(n-k)$$

Convolution Theorem:

If $Z[f(n)] = F(z)$ and $Z[g(n)] = G(z)$ then $Z[f(n) * g(n)] = Z[f(n)] \cdot Z[g(n)] = F(z)\cdot G(z)$

Note:

1) $Z[f(n) * g(n)] = F(z)\cdot G(z)$

$$f(n) * g(n) = Z^{-1}[F(z)\cdot G(z)]$$

$$Z^{-1}[F(z)] * Z^{-1}[G(z)] = Z^{-1}[F(z)\cdot G(z)] \quad \because Z^{-1}[F(z)] = f(n) \,\&\, Z^{-1}[G(z)] = g(n)$$

$Z^{-1}[F(z)\cdot G(z)] = Z^{-1}[F(z)] * Z^{-1}[G(z)]$

2) $1 + a + a^2 + a^3 + ... + a^n = \dfrac{a^{n+1} - 1}{a - 1}$

1. **Find inverse Z-transform of $\dfrac{z^2}{(z-a)^2}$ by using convolution theorem.**

Solution:

Given $Z^{-1}\left[\dfrac{z^2}{(z-a)^2}\right] = ?$

By convolution theorem

$Z^{-1}[F(z)\cdot G(z)] = Z^{-1}[F(z)] * Z^{-1}[G(z)]$

$$Z^{-1}\left[\frac{z^2}{(z-a)^2}\right] = Z^{-1}\left[\frac{z}{z-a}\cdot\frac{z}{z-a}\right]$$

$$= Z^{-1}\left[\frac{z}{z-a}\right] * Z^{-1}\left[\frac{z}{z-a}\right]$$

$$= a^n * a^n$$

$$= \sum_{k=0}^{n} a^k a^{n-k} \qquad \because f(n) * g(n) = \sum_{k=0}^{n} f(k)g(n-k)$$

$$= \sum_{k=0}^{n} a^k a^{-k} a^n$$

$$= a^n \sum_{k=0}^{n} 1$$

$$Z^{-1}\left[\frac{z^2}{(z-a)^2}\right] = a^n (n+1)\cdot 1 = (n+1)a^n$$

$$\boxed{Z^{-1}\left[\frac{z^2}{(z-a)^2}\right] = (n+1)a^n}$$

2. **By using convolution theorem, show that the inverse Z-transform of $\dfrac{z^2}{(z+a)(z+b)}$ is** $\dfrac{(-1)^n}{b-a}\left[b^{n+1} - a^{n+1}\right]$

Solution:

Given $Z^{-1}\left[\dfrac{z^2}{(z+a)(z+b)}\right] = ?$

By convolution theorem

$Z^{-1}[F(z)\cdot G(z)] = Z^{-1}[F(z)] * Z^{-1}[G(z)]$

$$Z^{-1}\left[\frac{z^2}{(z+a)(z+b)}\right] = Z^{-1}\left[\frac{z}{z+a}\cdot\frac{z}{z+b}\right]$$

$$= Z^{-1}\left[\frac{z}{z+a}\right] * Z^{-1}\left[\frac{z}{z+b}\right]$$

$$= (-a)^n * (-b)^n$$

$$= \sum_{k=0}^{n} (-a)^k (-b)^{n-k} \qquad \because f(n) * g(n) = \sum_{k=0}^{n} f(k)g(n-k)$$

$$= (-1)^n \sum_{k=0}^{n} a^k b^{-k} b^n$$

$$= (-1)^n b^n \sum_{k=0}^{n} \left[\frac{a}{b}\right]^k$$

$$= (-1)^n b^n \left[1 + \left(\frac{a}{b}\right) + \left(\frac{a}{b}\right)^2 + \left(\frac{a}{b}\right)^3 + \dots + \left(\frac{a}{b}\right)^n\right]$$

$$= (-1)^n b^n \left[\frac{\left(\frac{a}{b}\right)^{n+1} - 1}{\frac{a}{b} - 1}\right] = b^n \left[\frac{\frac{a^{n+1}}{b^{n+1}} - 1}{\frac{a}{b} - 1}\right] = b^n \left[\frac{\frac{a^{n+1} - b^{n+1}}{b^{n+1}}}{\frac{a-b}{b}}\right]$$

$$= (-1)^n b^n \left[\frac{a^{n+1} - b^{n+1}}{b^{n+1}} \times \frac{b}{a-b}\right] = (-1)^n \cancel{b^n} \left[\frac{a^{n+1} - b^{n+1}}{\cancel{b^n}\,\cancel{b}} \times \frac{\cancel{b}}{a-b}\right]$$

$$= (-1)^n \left[\frac{a^{n+1} - b^{n+1}}{a-b}\right]$$

$$\boxed{Z^{-1}\left[\frac{z^2}{(z+a)(z+b)}\right] = \frac{(-1)^n}{b-a}\left[b^{n+1} - a^{n+1}\right]}$$

3. **Find** $Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right]$ **using convolution theorem.**

Solution:

Given $Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right] = ?$

By convolution theorem

$$Z^{-1}\left[F(z) \cdot G(z)\right] = Z^{-1}\left[F(z)\right] * Z^{-1}\left[G(z)\right]$$

$$Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right] = Z^{-1}\left[\frac{z}{z-a} \cdot \frac{z}{z-b}\right]$$

$$= Z^{-1}\left[\frac{z}{z-a}\right] * Z^{-1}\left[\frac{z}{z-b}\right]$$

$$= (a)^n * (b)^n$$

$$= \sum_{k=0}^{n} (a)^k (b)^{n-k} \qquad \because f(n) * g(n) = \sum_{k=0}^{n} f(k) g(n-k)$$

$$= \sum_{k=0}^{n} a^k b^{-k} b^n$$

$$= b^n \sum_{k=0}^{n} \left[\frac{a}{b}\right]^k$$

$$= b^n \left[1 + \left(\frac{a}{b}\right) + \left(\frac{a}{b}\right)^2 + \left(\frac{a}{b}\right)^3 + \dots + \left(\frac{a}{b}\right)^n\right]$$

$$= b^n \left[\frac{\left(\frac{a}{b}\right)^{n+1} - 1}{\frac{a}{b} - 1} \right] = b^n \left[\frac{\frac{a^{n+1}}{b^{n+1}} - 1}{\frac{a}{b} - 1} \right] = b^n \left[\frac{\frac{a^{n+1} - b^{n+1}}{b^{n+1}}}{\frac{a-b}{b}} \right]$$

$$= b^n \left[\frac{a^{n+1} - b^{n+1}}{b^{n+1}} \times \frac{b}{a-b} \right] = (1)^n b^n \left[\frac{a^{n+1} - b^{n+1}}{b^n \, b} \times \frac{b}{a-b} \right]$$

$$= \frac{a^{n+1} - b^{n+1}}{a-b}$$

$$\boxed{Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right] = \frac{a^{n+1} - b^{n+1}}{a-b}}$$

4. **Using convolution theorem, find** $Z^{-1}\left[\frac{8z^2}{(2z-1)(4z+1)}\right]$

Solution:

Given $Z^{-1}\left[\frac{8z^2}{(2z-1)(4z+1)}\right] = ?$

By convolution theorem

$$Z^{-1}[F(z) \cdot G(z)] = Z^{-1}[F(z)] * Z^{-1}[G(z)]$$

$$Z^{-1}\left[\frac{8z^2}{(2z-1)(4z+1)}\right] = Z^{-1}\left[\frac{8z^2}{2\left(z-\frac{1}{2}\right) 4\left(z+\frac{1}{4}\right)}\right] = Z^{-1}\left[\frac{z}{\left(z-\frac{1}{2}\right)} \cdot \frac{z}{\left(z-\frac{1}{4}\right)}\right]$$

$$= Z^{-1}\left[\frac{z}{\left(z-\frac{1}{2}\right)}\right] * Z^{-1}\left[\frac{z}{\left(z-\frac{1}{4}\right)}\right]$$

$$= \left(\frac{1}{2}\right)^n * \left(\frac{1}{4}\right)^n$$

$$= \sum_{k=0}^{n} \left(\frac{1}{2}\right)^k \left(\frac{1}{4}\right)^{n-k} \qquad \because f(n) * g(n) = \sum_{k=0}^{n} f(k) g(n-k)$$

$$= \sum_{k=0}^{n} \left(\frac{1}{2}\right)^k \left(\frac{1}{4}\right)^n \left(\frac{1}{4}\right)^{-k}$$

$$= \left(\frac{1}{4}\right)^n \sum_{k=0}^{n} \left(\frac{1}{2}\right)^k (4)^k = \left(\frac{1}{4}\right)^n \sum_{k=0}^{n} \left(\frac{4}{2}\right)^k = \left(\frac{1}{4}\right)^n \sum_{k=0}^{n} (2)^k$$

$$= \left(\frac{1}{4}\right)^n \left[1 + 2 + 2^2 + 2^3 + \dots + 2^n\right]$$

$$= \left(\frac{1}{4}\right)^n \left[\frac{2^{n+1} - 1}{2 - 1}\right] \qquad \because 1 + a + a^2 + a^3 + \dots + a^n = \frac{a^{n+1} - 1}{a - 1}$$

$$\boxed{Z^{-1}\left[\frac{8z^2}{(2z-1)(4z+1)}\right] = \left(\frac{1}{4}\right)^n \left[2^{n+1} - 1\right]}$$

5. **Using convolution theorem find** $Z^{-1}\left[\frac{z^2}{(z-1)(z-3)}\right]$

Solution:

Given $Z^{-1}\left[\frac{z^2}{(z-1)(z-3)}\right] = ?$

By convolution theorem

$$Z^{-1}\left[F(z)\cdot G(z)\right] = Z^{-1}\left[F(z)\right] * Z^{-1}\left[G(z)\right]$$

$$Z^{-1}\left[\frac{z^2}{(z-1)(z-3)}\right] = Z^{-1}\left[\frac{z}{z-1}\cdot\frac{z}{z-3}\right]$$

$$= Z^{-1}\left[\frac{z}{z-1}\right] * Z^{-1}\left[\frac{z}{z-3}\right]$$

$$= (1)^n * (3)^n$$

$$= \sum_{k=0}^{n} (1)^k (3)^{n-k} \qquad \because f(n) * g(n) = \sum_{k=0}^{n} f(k) g(n-k)$$

$$= \sum_{k=0}^{n} 1^k 3^{-k} 3^n$$

$$= 3^n \sum_{k=0}^{n} \left(\frac{1}{3}\right)^k$$

$$= 3^n \left[1 + \left(\frac{1}{3}\right) + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + \dots + \left(\frac{1}{3}\right)^n\right]$$

$$= 3^n \left[\frac{\left(\frac{1}{3}\right)^{n+1} - 1}{\frac{1}{3} - 1}\right] = 3^n \left[\frac{\frac{1^{n+1}}{3^{n+1}} - 1}{\frac{1}{3} - 1}\right] = 3^n \left[\frac{\frac{1^{n+1} - 3^{n+1}}{3^{n+1}}}{\frac{1-3}{3}}\right]$$

$$= 3^n \left[\frac{1 - 3^{n+1}}{3^{n+1}} \times \frac{3}{1-3}\right] = 3^n \left[\frac{3^{n+1} - 3^{n+1}}{3^n\, 3} \times \frac{3}{-2}\right]$$

$$= \frac{-1}{2}\left(1 - 3^{n+1}\right)$$

$$Z^{-1}\left[\frac{z^2}{(z-1)(z-3)}\right] = \frac{-1}{2}\left(1 - 3^{n+1}\right)$$

Formation of Difference Equation:

1. Derive the difference equation from $y_n = (A + Bn)2^n$

Solution:

Given $y_n = (A + Bn)2^n$

$$y_n = A2^n + Bn2^n \qquad ------(1)$$

Replace n by $n+1$ in (1)

$$y_{n+1} = A2^{n+1} + B(n+1)2^{n+1}$$

$$y_{n+1} = 2A2^n + 2(n+1)B2^n \qquad -----(2)$$

Replace n by $n+2$ in (1)

$$y_{n+2} = A2^{n+2} + (n+2)B2^{n+2}$$

$$y_{n+2} = 4A2^n + 4(n+2)B2^n \qquad ----(3)$$

From (1), (2) and (3)

$$\begin{vmatrix} y_n & 1 & n \\ y_{n+1} & 2 & 2(n+1) \\ y_{n+2} & 4 & 4(n+2) \end{vmatrix} = 0$$

$$y_n\left[8(n+2) - 8(n+1)\right] - 1\left[4(n+2)y_{n+1} - 2(n+1)y_{n+2}\right] + n\left[4y_{n+1} - 2y_{n+2}\right] = 0$$

$$y_n\left[(8n + 16 - 8n - 8\right] - 1\left[(4n+8)y_{n+1} + (-2n-2)y_{n+2}\right] + 4ny_{n+1} - 2ny_{n+2} = 0$$

$$8y_n \cancel{-4ny_{n+1}} - 8y_{n+1} \cancel{+2ny_{n+2}} + 2y_{n+2} \cancel{+4ny_{n+1}} \cancel{-2ny_{n+2}} = 0$$

$$2y_{n+2} - 8y_{n+1} + 8y_n = 0$$

$$\boxed{y_{n+2} - 4y_{n+1} + 4y_n = 0}$$

2. **Derive the difference equation from** $u_n = a + b3^n$

Solution: $u_n = a + b3^n \quad ----(1)$

Replace n by $n+1$ in (1)

$$u_{n+1} = a + b3^{n+1}$$
$$u_{n+1} = a + 3b3^n \quad ----(2)$$

Replace n by $n+2$ in (1)

$$u_{n+2} = a + b3^{n+2}$$
$$u_{n+2} = a + 9b3^n \quad ----(3)$$

From (1), (2) and (3)

$$\begin{vmatrix} u_n & 1 & 1 \\ u_{n+1} & 1 & 3 \\ u_{n+2} & 1 & 9 \end{vmatrix} = 0$$

$$u_n(9-3) - 1(3u_{n+2} - 9u_{n+1}) + 1(u_{n+1} - u_{n+2}) = 0$$

$$6u_n - 3u_{n+2} + 9u_{n+1}) + u_{n+1} - u_{n+2} = 0$$

$$-4u_{n+2} + 10u_{n+1} + 6u_n = 0$$

$$\div(-2) \Rightarrow \boxed{2u_{n+2} - 5u_{n+1} - 3u_n = 0}$$

3. **Form the difference equation** $y_n = \cos\left(\frac{n\pi}{2}\right)$

Solution:

Given $y_n = \cos\left(\frac{n\pi}{2}\right) \quad ----(1)$

Replace n by $n+1$ in (1)

$$y_{n+1} = \cos\left(\frac{(n+1)\pi}{2}\right) = \cos\left(\frac{\pi}{2} + \frac{n\pi}{2}\right) = -\sin\left(\frac{n\pi}{2}\right) \quad ----(2)$$

Replace n by $n+2$ in (1)

$$y_{n+2} = \cos\left(\frac{(n+2)\pi}{2}\right) = \cos\left(\frac{2\pi}{2} + \frac{n\pi}{2}\right)$$

$$y_{n+2} = \cos\left(\pi + \frac{n\pi}{2}\right) = -\cos\left(\frac{n\pi}{2}\right)$$

$$y_{n+2} = -y_n \quad \text{from (1)}$$

$$\Rightarrow \boxed{y_{n+2} + y_n = 0}$$

Solutions of difference equation using Z-Transforms.

1. $Z\left[y_n\right] = Z\left[y(n)\right] = y(z)$

2. $Z\left[y_{n+1}\right] = Z\left[y(n+1)\right] = zy(z) - zy(0)$
3. $Z\left[y_{n+2}\right] = Z\left[y(n+2)\right] = z^2y(z) - z^2y(0) - zy(1)$
4. $Z\left[y_{n+3}\right] = Z\left[y(n+3)\right] = z_3y(z) - z_3y(0) - z_2y(1) - zy(2)$

1. **Solve using Z-transforms technique the difference equation** $y_{n+2} + 4y_{n+1} + 3y_n = 3^n$ **with**

$y_0 = 0, y_1 = 1$.

Solution: $y_{n+2} + 4y_{n+1} + 3y_n = 3^n$.

Taking Z-transform on both sides

$$Z\left[y_{n+2}\right] + 4Z\left[y_{n+1}\right] + 3Z\left[y_n\right] = Z\left(3^n\right)$$

$$\left(z^2y(z) - z^2y(0) - zy(1)\right) + 4\left[zy(z) - zy(0)\right] + 3y(z) = \frac{z}{z-3}$$

Given $y_0 = y(0) = 0, y_1 = y(1) = 1$

$$z^2y(z) - z + 4zy(z) + 3y(z) = \frac{z}{z-3}$$

$$(z^2 + 4z + 3)y(z) = \frac{z}{z-3} + z$$

$$(z^2 + 4z + 3)y(z) = \frac{z + z^2 - 3z}{z-3}$$

$$y(z) = \frac{z^2 - 2z}{(z-3)(z^2 + 4z + 3)}$$

$$y(z) = \frac{z(z-2)}{(z-3)(z+1)(z+3)}$$

By Partial Fraction,

$$\frac{y(z)}{z} = \frac{(z-2)}{(z-3)(z+1)(z+3)} \quad ----- (1)$$

Now $$\frac{(z-2)}{(z-3)(z+1)(z+3)} = \frac{A}{(z-3)} + \frac{B}{(z+1)} + \frac{C}{(z+3)}$$

$$z - 2 = A(z+1)(z+3) + B(z-3)(z+3) + C(z+1)(z-3)$$

Put z=3 $\Rightarrow 1 = 24A \Rightarrow A = \frac{1}{24}$

Put z= -1 $\Rightarrow -3 = -8B \Rightarrow B = \frac{3}{8}$

Put z = -3 $\Rightarrow -5 = 12C \Rightarrow C = \frac{-5}{12}$

$$\frac{(z-2)}{(z-3)(z+1)(z+3)} = \frac{1/24}{(z-3)} + \frac{3/8}{(z+1)} + \frac{-5/12}{(z+3)}$$

$$(1) \Rightarrow \frac{y(z)}{z} = \frac{1/24}{(z-3)} + \frac{3/8}{(z+1)} + \frac{-5/12}{(z+3)}$$

$$y(z) = \frac{1}{24}\frac{z}{(z-3)} + \frac{3}{8}\frac{z}{(z+1)} - \frac{5}{12}\frac{z}{(z+3)}$$

Taking Z^{-1} on both sides

$$Z^{-1}\left[y(z)\right] = \frac{1}{24}Z^{-1}\left(\frac{z}{z-3}\right) + \frac{3}{8}Z^{-1}\left(\frac{z}{z+1}\right) - \frac{5}{12}Z^{-1}\left(\frac{z}{z+3}\right)$$

	$y(n) = \frac{1}{24}(3)^n + \frac{3}{8}(-1)^n - \frac{5}{12}(-3)^n$ $\quad \because Z^{-1}\left[\frac{z}{z-a}\right] = a^n$
2.	**Solve $y_{n+2} - 3y_{n+1} - 10y_n = 0$, given $y_0 = 1$, $y_1 = 0$.** **Solution:** $y_{n+2} - 3y_{n+1} - 10y_n = 0$. Taking Z-transform on both sides $Z[y_{n+2}] - 3Z[y_{n+1}] - 10Z[y_n] = Z[0]$ $[z^2y(z) - z^2y(0) - zy(1)] - 3[zy(z) - zy(0)] - 10y(z) = 0$ Given $y_0 = y(0) = 1, y_1 = y(1) = 0$ $z^2y(z) - z^2 - 3zy(z) + 3z - 10y(z) = 0$ $(z^2 - 3z - 10)y(z) = z^2 - 3z$ $y(z) = \frac{z^2 - 3z}{(z^2 - 3z - 10)}$ $y(z) = \frac{z(z-3)}{(z+2)(z-5)}$ By Partial Fraction, $\frac{y(z)}{z} = \frac{(z-3)}{(z+2)(z-5)}$ - - - - - (1) Now $\frac{(z-3)}{(z+2)(z-5)} = \frac{A}{(z+2)} + \frac{B}{(z-5)}$ $z - 3 = A(z-5) + B(z+2)$ Put z= -2 $\Rightarrow -5 = -7A \Rightarrow A = \frac{5}{7}$ Put z= 5 $\Rightarrow 2 = 7B \Rightarrow B = \frac{2}{7}$ $\frac{(z-3)}{(z+2)(z-5)} = \frac{\frac{5}{7}}{(z+2)} + \frac{\frac{2}{7}}{(z-5)}$ $(1) \Rightarrow \frac{y(z)}{z} = \frac{\frac{5}{7}}{z+2} + \frac{\frac{2}{7}}{z-5}$ $y(z) = \frac{5}{7}\frac{z}{z+2} + \frac{2}{7}\frac{z}{z-5}$ Taking Z^{-1} on both sides $Z^{-1}[y(z)] = \frac{5}{7}Z^{-1}\left[\frac{z}{z+2}\right] + \frac{2}{7}Z^{-1}\left[\frac{z}{z-5}\right]$ $y(n) = \frac{5}{7}(-2)^n - \frac{2}{7}5^n$ $\quad \because Z^{-1}\left[\frac{z}{z-a}\right] = a^n$
3.	**Solve the equation** $y(n+3) - 3y(n+1) + 2y(n) = 0$ **given that** $y(0) = 4$, $y(1) = 0$ **and** $y(2) = 8$. **Solution:** $Z[y(n+3)] - 3Z[y(n+1)] + 2Z[y(n)] = Z[0]$ $[z^3y(z) - z^3y(0) - z^2y(1) - zy(2)] - 3[zy(z) - zy(0)] + 2y(z) = 0$ Given that $y(0) = 4$, $y(1) = 0$

$z^3 y(z) - 4z^3 - 8z - 3zy(z) + 12z + 2y(z) = 0$

$\left(z^3 - 3z + 2\right) y(z) = 4z^3 - 4z$

$$y(z) = \frac{4z^3 - 4z}{z^3 - 3z + 2}$$

$$y(z) = \frac{4z(z^2 - 1)}{(z-1)^2(z+2)}$$

$$y(z) = \frac{4z\,(z-1)(z+1)}{(z-1)^2(z+2)} \qquad \because a^2 - b^2 = (a+b)(a-b)$$

$$y(z) = \frac{4z(z+1)}{(z-1)(z+2)}$$

By Partial Fraction

$$\frac{y(z)}{z} = \frac{4(z+1)}{(z-1)(z+2)} \quad ------(1)$$

$$\frac{4(z+1)}{(z-1)(z+2)} = \frac{A}{z-1} + \frac{B}{z+2}$$

$4(z+1) = A(z+2) + B(z-1)$

Put $z = 1 \Rightarrow 8 = 3A \Rightarrow A = \frac{8}{3}$

Put $z = -2 \Rightarrow -4 = -3B \Rightarrow B = \frac{4}{3}$

$$\frac{y(z)}{z} = \frac{8/3}{z-1} + \frac{4/3}{z+2}$$

$$Z^{-1}\left[y(z)\right] = \frac{8}{3} Z^{-1}\left(\frac{z}{z-1}\right) + \frac{4}{3} Z^{-1}\left(\frac{z}{z+2}\right)$$

$$\boxed{y(n) = \frac{8}{3} + \frac{4}{3}(-2)^n} \qquad \because Z^{-1}\left(\frac{z}{z-a}\right) = a^n$$

4. **Using Z-transform solve** $y(n) + 3y(n-1) - 4y(n-2) = 0, n \geq 2$ **given that** $y(0) = 3$ and $y(1) = -2$

Solution:

Given $y(n) + 3y(n-1) - 4y(n-2) = 0, n \geq 2$

Replace n by $n+2$, weget

$y(n+2) + 3y(n+1) - 4y(n) = 0$

Taking Z transforms on both sides

$Z\left[y(n+2)\right] + 3Z\left[y(n+1)\right] - 4Z\left[y(n)\right] = Z\left[0\right]$

$\left(z^2 y(z) - z^2 y(0) - zy(1)\right) + 3\left[zy(z) - zy(0)\right] - 4y(z) = 0$

Given that $y(0) = 3$ and $y(1) = -2$

$\left(z^2 y(z) - 3z^2 + 2z\right) + 3\left[zy(z) - 3z\right] - 4y(z) = 0$

$\left(z^2 + 3z - 4\right) y(z) - 3z^2 + 2z - 9z = 0$

$\left(z^2 + 3z - 4\right) y(z) = 3z^2 + 7z$

$$y(z) = \frac{3z^2 + 7z}{z^2 + 3z - 4}$$

By Partial Fraction

$$\frac{y(z)}{z} = \frac{3z + 7}{z^2 + 3z - 4} = \frac{3z + 7}{(z+4)(z-1)}$$

	Now, $\frac{3z+7}{(z+4)(z-1)} = \frac{A}{z+4} + \frac{B}{z-1}$ $3z+7 = A(z-1) + B(z+4)$ Put $z = 1 \Rightarrow 10 = 5B \Rightarrow B = 2$ Put $z = -4 \Rightarrow -5 = -5A \Rightarrow A = 1$ $\frac{y(z)}{z} = \frac{1}{z+4} + \frac{2}{z-1}$ $y(z) = \frac{z}{z+4} + 2\frac{z}{z-1}$ $Z^{-1}[y(z)] = Z^{-1}\left(\frac{z}{z+4}\right) + 2Z^{-1}\left(\frac{z}{z-1}\right)$ $\boxed{y(n) = (-4)^n + 2(1)^n = 2 + (-4)^n}$ $\quad \because Z^{-1}\left(\frac{z}{z-a}\right) = a^n$
5.	**Solve using Z-transforms technique the difference equation $u_{n+2} + 6u_{n+1} + 9u_n = 2^n$ with** $u_0 = u_1 = 0$. Solution: $u_{n+2} + 6u_{n+1} + 9u_n = 2^n$ Assume u=y $y_{n+2} + 6y_{n+1} + 9y_n = 2^n \quad ; y_0 = y_1 = 0$ Taking Z-transform on both sides $Z[y_{n+2}] + 6Z[y_{n+1}] + 9Z[y_n] = Z(2^n)$ $\left(z^2y(z) - z^2y(0) - zy(1)\right) + 6[zy(z) - zy(0)] + 9y(z) = \frac{z}{z-2}$ Given $y_0 = y(0) = 0$; $y_1 = y(1) = 0$ $z^2y(z) + 6zy(z) + 9y(z) = \frac{z}{z-2}$ $(z^2 + 6z + 9)y(z) = \frac{z}{z-2}$ $y(z) = \frac{z}{(z-2)(z^2+6z+9)}$ $y(z) = \frac{z}{(z-2)(z+3)^2}$ By Partial Fraction, $\frac{y(z)}{z} = \frac{1}{(z-2)(z+3)^2} \quad ------(1)$ Now $\frac{1}{(z-2)(z+3)^2} = \frac{A}{(z-2)} + \frac{B}{(z+3)} + \frac{C}{(z+3)^2}$ $1 = A(z+3)^2 + B(z-2)(z+3) + C(z-2)$ Put $z = 2 \Rightarrow 1 = 25A \Rightarrow A = \frac{1}{25}$ Put $z = -3 \Rightarrow 1 = -5C \Rightarrow C = \frac{-1}{5}$ Equating co-efft. of z^2 on both sides $\Rightarrow A + B = 0 \Rightarrow B = -A \Rightarrow B = -\frac{1}{25}$

$$\frac{y(z)}{z} = \frac{\frac{1}{25}}{(z-2)} + \frac{\frac{-1}{25}}{(z+3)} + \frac{\frac{-1}{5}}{(z+3)^2}$$

Taking Z^{-1} on both sides

$$Z^{-1}[y(z)] = \frac{1}{25} Z^{-1}\left[\frac{z}{z-2}\right] - \frac{1}{25} Z^{-1}\left[\frac{z}{z+3}\right] - \frac{1}{5} Z^{-1}\left[\frac{z}{(z+3)^2}\right]$$

$$\boxed{y(n) = \frac{1}{25}(2)^n - \frac{1}{25}(-3)^n - \frac{1}{5}n(-3)^{n-1}} \quad \because Z^{-1}\left[\frac{z}{(z-a)^2}\right] = na^{n-1} \ \& \ Z^{-1}\left[\frac{z}{z-a}\right] = a^n$$

$$\boxed{u(n) = \frac{1}{25}(2)^n - \frac{1}{25}(-3)^n - \frac{1}{5}n(-3)^{n-1}} \quad \because u = y$$

6. **Using Z-transform method solve $y(k+2) + y(k) = 2$ given that $y_0 = y_1 = 0$.**

Solution:

Given $y(k+2) + y(k) = 2$; $y_0 = y_1 = 0$.

Assume $k=n$

$$y(n+2) + y(n) = 2$$

Taking Z-transform on both sides

$$Z[y(n+2)] + Z[y(n)] = 2Z[1]$$

$$z^2 y(z) - z^2 y(0) - zy(1) + y(z) = 2\frac{z}{z-1}$$

Given that $y_0 = y_1 = 0$.

$$(z^2+1)y(z) = \frac{2z}{z-1}$$

$$y(z) = \frac{2z}{(z-1)(z^2+1)}$$

$$\frac{y(z)}{z} = \frac{2}{(z-1)(z^2+1)} \quad ------(1)$$

By partial fraction

Now, $$\frac{2}{(z-1)(z^2+1)} = \frac{A}{z-1} + \frac{B}{z^2+1} + \frac{Cz}{z^2+1}$$

$$2 = A(z^2+1) + B(z-1) + Cz(z-1)$$

Put $z=1 \Rightarrow 2 = 2A \Rightarrow A = 1$

Put $z=0 \Rightarrow 2 = A - B \Rightarrow B = A - 2 \Rightarrow B = -1$

Equating co-efft. of z^2 on both sides $\Rightarrow 0 = A + C \Rightarrow C = -A \Rightarrow C = -1$

$$(1) \Rightarrow \frac{y(z)}{z} = \frac{1}{z-1} + \frac{-1}{z^2+1} + \frac{-z}{z^2+1}$$

$$y(z) = \frac{z}{z-1} - \frac{z}{z^2+1} - \frac{z^2}{z^2+1}$$

Taking Z^{-1} on both sides

$$Z^{-1}[y(z)] = Z^{-1}\left[\frac{z}{z-1}\right] - Z^{-1}\left[\frac{z}{z^2+1}\right] - Z^{-1}\left[\frac{z^2}{z^2+1}\right]$$

$$y(n) = (1)^n - 1^n \sin\frac{n\pi}{2} - 1^n \cos\frac{n\pi}{2}$$

$$\boxed{y(n) = 1 - \sin\frac{n\pi}{2} - \cos\frac{n\pi}{2}}$$

	$y(k) = 1 - \sin\frac{k\pi}{2} - \cos\frac{k\pi}{2}$ $\because Z^{-1}\left[\frac{z}{z^2+a^2}\right] = a^n \sin\frac{n\pi}{2}$ & $Z^{-1}\left[\frac{z^2}{z^2+a^2}\right] = a^n \cos\frac{n\pi}{2}$ here $a = 1$

Problems based on Z-Transforms:

1. **Find** $Z[\cos n\theta]$, $Z[\sin n\theta]$ **and hence find i)** $Z\left[\cos\frac{n\pi}{2}\right]$, ii) $Z\left[\sin\frac{n\pi}{2}\right]$

iii) $Z\left[r^n \cos n\theta\right]$, iv) $Z\left[r^n \sin n\theta\right]$

Solution:

We know that $e^{in\theta} = \cos n\theta + \sin n\theta$

$\cos n\theta$ = real part of $e^{in\theta}$ & $\sin n\theta$ = imaginary part of $e^{in\theta}$

and $Z\left[a^n\right] = \frac{z}{z-a}$

$$Z\left[e^{in\theta}\right] = Z\left[\left(e^{i\theta}\right)^n\right] = \frac{z}{z-e^{i\theta}}$$

$$= \frac{z}{z-(\cos\theta + i\sin\theta)}$$

$$= \frac{z}{(z-\cos\theta) - i\sin\theta} \times \frac{(z-\cos\theta)+i\sin\theta}{(z-\cos\theta)+i\sin\theta}$$

$$Z\left[e^{in\theta}\right] = \frac{z(z-\cos\theta)+i\sin\theta}{(z-\cos\theta)^2 - i^2\sin^2\theta} \quad \because (a+b)(a-b) = a^2 - b^2$$

$$Z[\cos n\theta + i\sin n\theta] = \frac{z(z-\cos\theta)+iz\sin\theta}{z^2 - 2z\cos\theta + \cos^2\theta + \sin^2\theta} \quad \because i^2 = -1$$

$$Z[\cos n\theta] + iZ[\sin n\theta] = \frac{z(z-\cos\theta)}{z^2 - 2z\cos\theta + 1} + i\frac{z\sin\theta}{z^2 - 2z\cos\theta + 1} \quad \because \cos^2\theta + \sin^2\theta = 1$$

Equating co-efft. Of real and img parts on both sides

$$Z[\cos n\theta] = \frac{z(z-\cos\theta)}{z^2 - 2z\cos\theta + 1} \; ; \; Z[\sin n\theta] = \frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}$$

Deduction:

We know that

$$Z[\cos n\theta] = \frac{z(z-\cos\theta)}{z^2 - 2z\cos\theta + 1}$$

i) $Z\left[\cos\frac{n\pi}{2}\right] = Z[\cos n\theta]_{\theta \to \frac{\pi}{2}} = \frac{z\left[z-\cos\frac{\pi}{2}\right]}{z^2 - 2z\cos\frac{\pi}{2} + 1}$

$$Z\left[\cos\frac{n\pi}{2}\right] = \frac{z^2}{z^2+1} \quad \because \cos\frac{\pi}{2} = 0$$

$$Z[\sin n\theta] = \frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}$$

ii) $Z\left[\sin\frac{n\pi}{2}\right] = Z[\sin n\theta]_{\theta \to \frac{\pi}{2}} = \frac{z\sin\frac{\pi}{2}}{z^2 - 2z\cos\frac{\pi}{2} + 1}$

$$\therefore Z\left[\sin\frac{n\pi}{2}\right] = \frac{z}{z^2+1} \quad \because \cos\frac{\pi}{2} = 0 \;\&\; \sin\frac{\pi}{2} = 1$$

We know that

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$Z\left(a^n f(n)\right) = Z[f(n)]_{z\to \frac{z}{a}}$

iii) $Z\left(r^n \cos n\theta\right) = \left(Z[\cos n\theta]\right)_{z\to \frac{z}{r}}$

$$= \left(\frac{z(z-\cos\theta)}{z^2 - 2z\cos\theta + 1}\right)_{z\to\frac{z}{r}}$$

$$= \frac{\frac{z}{r}\left(\frac{z}{r} - \cos\theta\right)}{\left(\frac{z}{r}\right)^2 - \frac{2z}{r}\cos\theta + 1}$$

$$= \frac{\frac{z}{r}\left(\frac{z - r\cos\theta}{r}\right)}{\frac{z^2 - 2zr\cos\theta + r^2}{r^2}}$$

$$Z\left(r^n \cos n\theta\right) = \frac{z(z - r\cos\theta)}{z^2 - 2zr\cos\theta + r^2}$$

iv) $Z\left(r^n \sin n\theta\right) = \left(Z\{\sin n\theta\}\right)_{z\to\frac{z}{r}} = \frac{\frac{z}{r}\sin\theta}{\frac{z^2}{r^2} - 2\frac{z}{r}\cos\theta + r^2} = \frac{\frac{z}{r}\sin\theta}{\frac{z^2 - 2zr\cos\theta + r^2}{r^2}}$

$$Z\left(r^n \sin n\theta\right) = \frac{zr\sin\theta}{z^2 - 2zr\cos\theta + r^2}$$

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<tr><td>**2.**</td><td>

Find the Z-transform of $\frac{1}{n(n+1)}$, *for* $n \geq 1$

Solution

$Z\left[\frac{1}{n(n+1)}\right] = ?$

By partial Fraction:

$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$$

$1 = A(n+1) + Bn$

Put $n = -1;\ 1 = -B \Rightarrow B = -1$

Put $n = 0;\quad A = 1$

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

$Z\left[\frac{1}{n(n+1)}\right] = Z\left[\frac{1}{n}\right] - Z\left[\frac{1}{n+1}\right]$ ------(1)

Now, we know that

$$Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}$$

$Z\left(\frac{1}{n}\right) = \sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{1}{z}\right)^n \quad \because n > 0$

$$= \frac{1}{z} + \frac{1}{2}\left(\frac{1}{z}\right)^2 + \frac{1}{3}\left(\frac{1}{z}\right)^3 + \ldots.$$

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$$= x + \frac{x^2}{2} + \frac{x^3}{3} + ... \quad \text{here } \frac{1}{z} = x$$

$$= -\log(1-x)$$

$$Z\left[\frac{1}{n}\right] = -\log\left(1-\frac{1}{z}\right) = -\log\left(\frac{z-1}{z}\right) = \log\left(\frac{z}{z-1}\right)$$

$$Z\left[\frac{1}{n}\right] = \log\left(\frac{z}{z-1}\right)$$

$$Z\left[\frac{1}{n+1}\right] = \sum_{n=0}^{\infty} \frac{1}{n+1}\left(\frac{1}{z}\right)^n$$

$$= 1 + \frac{1}{2}\left(\frac{1}{z}\right) + \frac{1}{3}\left(\frac{1}{z}\right)^2 + ...$$

$$= z\left[\frac{1}{z} + \frac{1}{2}\left(\frac{1}{z}\right)^2 + \frac{1}{3}\left(\frac{1}{z}\right)^3 + ..\right]$$

$$= z\left[-\log\left(1-\frac{1}{z}\right)\right] = -z\log\left(\frac{z-1}{z}\right)$$

$$Z\left[\frac{1}{n+1}\right] = z\log\left(\frac{z}{z-1}\right)$$

$$(1) \Rightarrow \quad Z\left[\frac{1}{n(n+1)}\right] = \log\left(\frac{z}{z-1}\right) + z\log\left(\frac{z}{z-1}\right)$$

$$\therefore Z\left[\frac{1}{n(n+1)}\right] = (z+1)\log\left(\frac{z}{z-1}\right)$$

3. **Find** $Z\left[n(n-1)(n-2)\right]$.

Solution:

$$Z\left[n(n-1)(n-2)\right] = Z\left[(n^2-n)(n-2)\right] = Z\left[n^3 - 2n^2 - n^2 + 2n\right] = Z\left[n^3 - 3n^2 + 2n\right]$$

$$Z\left[n(n-1)(n-2)\right] = Z\left[n^3\right] - 3Z\left[n^2\right] + 2Z[n] \quad ----(1)$$

We know that

$$Z[f(n)] = \sum_{n=0}^{\infty} f(n)z^{-n}$$

$$Z[n] = \sum_{n=0}^{\infty} n\left(\frac{1}{z}\right)^n$$

$$= 0 + 1\left(\frac{1}{z}\right)^1 + 2\left(\frac{1}{z}\right)^2 + 3\left(\frac{1}{z}\right)^3 + ...$$

$$= x + 2x^2 + 3x^3 + ...$$

$$= x(1 + 2x + 3x^2 + ...) = x(1-x)^{-2} = \left(\frac{1}{z}\right)\left(1-\frac{1}{z}\right)^{-2}$$

$$= \frac{1}{z}\left(\frac{z-1}{z}\right)^{-2} = \frac{1}{z}\left(\frac{z}{z-1}\right)^2 = \frac{1}{z}\left(\frac{z^2}{(z-1)^2}\right)$$

$$Z[n] = \frac{z}{(z-1)^2}$$

We know that $Z[nf(n)] = -z\frac{d}{dz}\{Z[f(n)]\}$

$$Z[n^2] = -z\frac{d}{dz}\{Z[n]\}$$

$$= -z\frac{d}{dz}\left[\frac{z}{(z-1)^2}\right]$$

$$= -z\left[\frac{(z-1)^2(1) - z[2(z-1)]}{(z-1)^4}\right]$$

$$= -z\left[\frac{(z-1)(z-1-2z)}{(z-1)^4}\right]$$

$$= -z\left[\frac{-1-z}{(z-1)^3}\right]$$

$$Z[n^2] = \frac{z+z^2}{(z-1)^3}$$

$$Z[n^3] = Z[n\,n^2] = -z\frac{d}{dz}\{Z[n^2]\}$$

$$= -z\frac{d}{dz}\left[\frac{z+z^2}{(z-1)^3}\right]$$

$$= z\left[\frac{(z-1)^3(2z+1) - (z^2+z)3(z-1)^2(1-0)}{(z-1)^6}\right]$$

$$= -z\left[\frac{(z-1)^2[(z-1)(2z+1) - 3(z^2+z)]}{(z-1)^6}\right]$$

$$= z\left[\frac{2z^2-2z+z-1-3z^2-3z}{(z-1)^4}\right]$$

$$= -z\left[\frac{-z^2-4z-1}{(z-1)^4}\right]$$

$$Z[n^3] = \frac{z(z^2+4z+1)}{(z-1)^4}$$

$$(1) \Rightarrow \quad Z[n(n-1)(n-2)] = \frac{z(z^2+4z+1)}{(z-1)^4} - 3\frac{z+z^2}{(z-1)^3} + 2\frac{z}{(z-1)^2}$$

4. **If** $U(z) = \frac{2z^2+5z+14}{(z-1)^4}$**, evaluate** u_2 and u_3

Solution:

Given $U(z) = F(z) = \frac{2z^2+5z+14}{(z-1)^4}$

We know that

$$u_0 = f(0) = \lim_{z\to\infty} F(z) = \lim_{z\to\infty}\frac{2z^2+5z+14}{(z-1)^4} = \lim_{z\to\infty}\frac{z^2\left[2+\frac{5}{z}+\frac{14}{z^2}\right]}{z^4\left[1-\frac{1}{z}\right]^2}$$

$$u_0 = f(0) = 0 \quad \because \frac{1}{\infty} = 0$$

	$u_1 = f(1) = \lim_{z\to\infty}\left[zF(z) - zf(0)\right]$ $= \lim_{z\to\infty}\left[\frac{z(2z^2+5z+14)}{(z-1)^4} - z(0)\right]$ $= \lim_{z\to\infty}\left[\frac{z^3\left(2+\frac{5}{z}+\frac{14}{z^2}\right)}{z^4\left(1-\frac{1}{z}\right)^4} - 0\right]$ $u_1 = f(1) = 0 \quad \because \frac{1}{\infty} = 0$ $u_2 = f(2) = \lim_{z\to\infty}\left[z^2F(z) - z^2 f(0) - zf(1)\right]$ $= \lim_{z\to\infty}\left[\frac{z^2(2z^2+5z+14)}{(z-1)^4} - z^2(0) - z(0)\right]$ $= \lim_{z\to\infty}\left[\frac{z^4\left(2+\frac{5}{z}+\frac{14}{z^2}\right)}{z^4\left(1-\frac{1}{z}\right)^4}\right] = \frac{2+0+0}{(1-0)^4} = 2$ $u_2 = f(2) = 2$ $u_3 = f(3) = \lim_{z\to\infty}\left[z^3F(z) - z^3 f(0) - z^2 f(1) - zf(2)\right]$ $= \lim_{z\to\infty}\left[\frac{z^3(2z^2+5z+14)}{(z-1)^4} - z^3(0) - z^2(0) - z(2)\right]$ $= \lim_{z\to\infty}\left[\frac{z^3(2z^2+5z+14)}{(z-1)^4} - 2z\right]$ $= \lim_{z\to\infty} z^3\left[\frac{(2z^2+5z+14)}{(z-1)^4} - \frac{2}{z^2}\right]$ $= \lim_{z\to\infty} z^3\left[\frac{z^2(2z^2+5z+14) - 2(z-1)^4}{z^2(z-1)^4}\right] \quad \because (a-b)^4 = a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$ $= \lim_{z\to\infty} z^3\left[\frac{(2z^4+5z^3+14z^2) - 2(z^4-4z^3+6z^2-4z+1)}{z^2(z-1)^4}\right]$ $= \lim_{z\to\infty} z^3\left[\frac{2z^4+5z^3+14z^2-2z^4+8z^3-12z^2+8z-2}{z^2(z-1)^4}\right]$ $= \lim_{z\to\infty} z^3\left[\frac{13z^3+2z^2+8z-2}{z^2(z-1)^4}\right]$ $= \lim_{z\to\infty}\frac{z^6\left(13+\frac{2}{z}+\frac{8}{z^2}-\frac{2}{z^3}\right)}{z^6\left(1-\frac{1}{z}\right)^4} = \lim_{z\to\infty}\frac{13+\frac{2}{z}+\frac{8}{z^2}-\frac{2}{z^3}}{\left(1-\frac{1}{z}\right)^4} = \frac{13+0+0-0}{(1-0)^4}$ $u_3 = f(3) = 13$
5.	**State and prove initial and final value theorem of Z-transform.** **Initial value theorem:** **If** $Z\left[f(n)\right] = F(z)$ **then** $f(0) = \lim_{z\to\infty} F(z)$ **Proof:**

We know that

$$Z\left[f(n)\right] = \sum_{n=0}^{\infty} f(n) z^{-n}$$

$$\lim_{z\to\infty} F(z) = \lim_{z\to\infty} \sum_{n=0}^{\infty} f(n)\left(\frac{1}{z}\right)^{n}$$

$$= \lim_{z\to\infty}\left[f(0)\left(\frac{1}{z}\right)^{0} + f(1)\left(\frac{1}{z}\right)^{1} + f(2)\left(\frac{1}{z}\right)^{2} + \ldots\right]$$

$$\lim_{z\to\infty} F(z) = f(0) \quad \because \frac{1}{\infty} = 0$$

Final value theorem:

If $Z\left[f(n)\right] = F(z)$ **then** $\lim_{n\to\infty} f(n) = \lim_{z\to 1}(z-1)F(z)$

Proof:

$$Z\left[f(n)\right] = \sum_{n=0}^{\infty} f(n) z^{-n} \quad ------(1)$$

$$Z\left[f(n+1)\right] = \sum_{n=0}^{\infty} f(n+1) z^{-n} \quad ------(2)$$

$(1)-(2) \Rightarrow$

$$Z\left[f(n+1)\right] - Z\left[f(n)\right] = \sum_{n=0}^{\infty} f(n+1) z^{-n} - \sum_{n=0}^{\infty} f(n) z^{-n}$$

$$\left[zF(z) - zf(0)\right] - F(z) = \sum_{n=0}^{\infty}\left[f(n+1) - f(n)\right] z^{-n}$$

$$\lim_{z\to 1}\left[(z-1)F(z) - zf(0)\right] = \lim_{z\to 1} \sum_{n=0}^{\infty}\left[f(n+1) - f(n)\right] z^{-n}$$

$$\lim_{z\to 1}\left[(z-1)F(z)\right] - f(0) = \sum_{n=0}^{\infty}\left[f(n+1) - f(n)\right]$$

$$\lim_{z\to 1}\left[(z-1)F(z)\right] - f(0) = \left[\cancel{f(1)} - f(0)\right] + \left[\cancel{f(2)} - \cancel{f(1)}\right] + \ldots + \left[f(n+1) - \cancel{f(n)}\right] + \ldots\infty$$

$$\lim_{z\to 1}\left[(z-1)F(z)\right] - \cancel{f(0)} = -\cancel{f(0)} + f(n+1) + \ldots\infty$$

$$\lim_{z\to 1}\left[(z-1)F(z)\right] = \lim_{n\to\infty} f(n) \quad \because f(n+1) = f(n) \text{ when } n \to \infty$$

Hence proved