Gram Schmidt Orthogonalisation Process

Theorem:

Every finite dimensional inner product space has an orthonormal basis.(Gram Schmidt Orthogonalization Process)

PROBLEMS

Apply Gram-Schmidt process to construct an orthonormal basis for $V_3(R)$ with the standard inner product for the basis (v_1,v_2,v_3), Where $v_1 = (1,0,1)$; $v_2 = (1,3,1)$ and $v_3 = (3,2,1)$.

Sol: The first vector in the orthogonal basis is

$$w_1 = v_1 = (1,0,1)$$

The formula for the second vector in the orthogonal basis is $W_2 = v_2 - \frac{(v_2,w_1)}{\|w_1\|^2} w_1$

The quantities that we need for this step are

$$(v_2,w_2)=((1,3,1),(1,0,1))$$

$$=1 + 0 +1 = 2$$

$$\|w_1\|^2=(w_1,w_2)=1^2 +0^2 +1^2 =2$$

The quantities that we need for this step are

$$(v_2,w_2)= ((1,3,1),(1,0,1))$$

$$=1 + 0 +1 = 2$$

$$\|w_1\|^2=(w_1,w_2)=1^2 +0^2 +1^2 =2 .$$

Therefore the second vector is

$w_2 = (1,3,1) - \frac{2}{2}(1,0,1)$

$= (1,3,1) - (1,0,1) \Rightarrow (0,3,0).$

The formula for the third (and final) vector in the orthogonal basis is

$$w_3 = v_3 - \frac{(v_3,w_1)}{\|w_1\|^2} w_1 - \frac{(v_3,w_1)}{\|w_2\|^2} w_2$$

The quantities that we need for this steps are

$\|w_2\|^2 = (w_2,w_2) = 0^2 + 3^2 + 0^2 = 9.$

$(v_3,w_1) = ((3,2,1),(1,0,1)) = 3 + 0 + 1 = 4$

$(v_3,w_2) = ((3,2,1),(0,3,0)) = 0 + 6 + 0 = 6$

Therefore the third vector is

$W_3 = (3,2,1) - \frac{4}{2}(1,0,1) - \frac{6}{9}(0,3,0)$

$= (3,2,1) - 2(1,0,1) - \frac{2}{3}(0,3,0)$

$= (1,0,-1).$

$\|w_3\|^2 = (w_3,w_3) = 1^2 + 0^2 + (-1)^2 = 2.$

The orthogonal basis is

$\{(1,0,1),(0,3,0),(1,0,-1)\}$

The orthonormal basis is

$$\beta = \{b_1, b_2, b_3\},$$

Where $b_1 = \frac{(w_1)}{\|w_1\|}$, $b_2 = \frac{(w_2)}{\|w_2\|}$, $b_3 = \frac{(w_3)}{\|w_3\|}$,

$\|w_1\|^2 = 2 \Rightarrow \|w_1\| = \sqrt{2}$

$\|w_2\|^2 = 9 \Rightarrow \|w_2\| = 3$

$\|w_3\|^2 = 2 \Rightarrow \|w_3\| = \sqrt{2}$

$$b_1 = \frac{(w_1)}{\|w_1\|} = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$$

$$b_2 = \frac{(w_2)}{\|w_2\|} = (0,1,0)$$

$$b_3 = \frac{(w_3)}{\|w_3\|} = \left(\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right)$$

Therefore the orthonormal basis is

$$\beta = \{b_1, b_2, b_3\}$$

$$\beta = \left\{\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), (0,1,0)\left(\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right)\right\}.$$

Definition

Let V be a finite dimensional inner product space and let T be a linear operator on V. Then there exist a unique function $T^*: V \to V$ such that $\langle T(x), y\rangle$, $\langle x, T^*(y)\rangle$ for all $x, y \in V$. The linear operator T^* is called adjoint of operator T.

Theorem 3.14: Let T be a linear functional on a finite dimensional inner product space V. Then there exists a unique vector $y \in V$ such that $g(x) = \langle x, y\rangle$ for every $x \in V$.

Proof: Let $\beta = \{v_1, v_2, \dots, v_n\}$ be an orthonormal basis of V.
Let $y = \overline{g(v_1)}v_1 + \overline{g(v_2)}v_2 + \cdots + \overline{g(v_n)}v_n$

Define $h: V \to F$ by $h(x) = \langle x, y\rangle$ for every $y \in V$.

It is clear that h is linear.

Then for $i = 1,2, \dots \dots, n$,

$h(v_i) = \langle v_i, y\rangle = \langle v_i, \overline{g(v_1)}v_1 + \overline{g(v_2)}v_2 + \cdots + \overline{g(v_n)}v_n\rangle$

$= \langle v_i, \overline{g(v_i)}v_i\rangle [\because \langle v_i, v_j\rangle = 0 \text{ for } i \neq j]$

$= g(v_i)\langle v_i, v_i\rangle = g(v_i)\|v_i\|^2 \; [\because \|v_i\|^2 = 1] \; \therefore h(v_i) = g(v_i)$

This is true for each $v_i, i = 1,2, \dots, n$

$$\therefore h = g$$

We have to prove the uniqueness.

Now suppose that y' is another vector in V for which

$$g(x) = \langle x, y'\rangle \text{ for each } x \in V$$

Then

$$\langle x, y\rangle = \langle x, y'\rangle$$

$$\Rightarrow (x, y) - \langle x, y'\rangle = 0 \Rightarrow (x, y - y') = 0 \Rightarrow y - y' = 0 \Rightarrow y = y'$$

$$\therefore \text{y is unique}$$

Let T be a linear operator on a finite dimensional inner prods then there exists a unique linear operator T' on V such that

$$\langle \boldsymbol{T}(\boldsymbol{x}), \boldsymbol{y}\rangle = \langle \boldsymbol{x}, \boldsymbol{T}^*(\boldsymbol{y})\rangle \textbf{ for every } \boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{V}.$$

Proof: Let y be an arbitrary but fixed element of V.

g: V →= F by $g(x) = \langle T(x), y\rangle$ for every $y \in V$.

First we prove that g is linear.

La $x_1, x_2 \in V$ and $\alpha \in F$.

$$\begin{aligned}(i) g(x_1 + x_2) &= (T(x_1 + x_2), y) \\ &= \langle T(x_1) + T(x_2), y\rangle [\because T \text{ is linear }] \\ &= \langle T(x_1), y\rangle + \langle T(x_2), y\rangle \\ &= g(x_1) + g(x_2)\end{aligned}$$

$$\begin{aligned}(ii) g(\alpha x_1) = &\ \langle T(\alpha x_1), y\rangle \\ = &\ \langle \alpha T(x_1), y\rangle [\because T \text{ is linear }] \\ = &\ \alpha \langle T(x_1), y\rangle \\ = &\ \alpha g(x_1)\end{aligned}$$

Therefore g is a linear transformation on V.

By Theorem 3.14, There exists a unique vector $y' \in V$ such that

Define $T^*: V \to V$ by $T^*(y) = y'$ for $y \in V$.

Therefore $\langle T(x), y\rangle = \langle x, T^*(y)\rangle$ for each $x \in V$.

We have to prove that T^* is linear

Let $y_1, y_2 \in V$ and $\alpha \in F$.

(i)
$$\begin{aligned}\langle x, T^*(y_1 + y_2)\rangle &= \langle T(x), y_1 + y_2\rangle \\ &= \langle T(x), y_1\rangle + \langle T(x), y_2\rangle \\ &= \langle x, T^*(y_1)\rangle + \langle x, T^*(y_2)\rangle\end{aligned}$$

Since x is arbitrary,

$$\langle T^*(y_1 + y_2) = T^*(y_1) + T^*(y_2)$$

(ii) $\langle x, T^*(\alpha y_1)\rangle == \langle T(x), \alpha y_1\rangle$

$$\begin{aligned}&= \bar{\alpha}\langle T(x), y_1\rangle \\ &= \bar{\alpha}\langle x, T^*(y_1)\rangle \\ &= \langle x, \alpha T^*(y_1)\rangle\end{aligned}$$

Since x is arbitrary,

$$T^*(\alpha y_1) = \alpha T^*(y_1)$$

Therefore T^* is linear.

Finally, we need to show that T^* is unique. Suppose that $U: V \to V$.

is linear and that it satisfies $\langle T(x), y\rangle = \langle x, U(y)\rangle$ for all $x, y \in V$. Then

$\langle x, T^*(y)\rangle = \langle x, U(y)\rangle$ for all $x, y \in V$, so

$$T^* = U.$$

Theorem 3.16: Let V be a finite-dimensional inner product space, and let β be an orthonormal basis for V. If T is a linear operator on V, then $[T^*]_\beta = [T]_\beta^*$

Proof: Let $A = [T^*]_\beta$ and $B = [T]_\beta^*$ and, $\beta = \{v_1, v_2, \ldots, v_n\}$ be an orthonormal basis of V. Then

$$\begin{aligned} B_{ij} &= \langle T^*(v_j), v_i\rangle \\ &= \overline{\langle v_i, T^*(v_j)\rangle} \\ &= \overline{\langle T(v_i), v_j\rangle} \\ &= \bar{A}_{ji} \\ &= A_{ij}^* \end{aligned}$$

Thus $B = A^*$

Theorem 3.17: Let T and U be linear operators on a finite dimensional inner product space V and $\alpha \epsilon F$. Then

(i) $(T + U)^* = T^* + U^*$

(ii) $(\alpha T)^* = \bar{\alpha} T^*$

(iii) $(TU)^* = U^* T^*$

(iv) $(T^*)^* = T$

(v) $I^* = I$

Proof

(i) Let $x, y \in V$

$$\begin{aligned} \langle (T + U)x, y\rangle = & \langle T(x) + U(x), y\rangle \\ & = \langle T(x), y\rangle + \langle U(x), y\rangle \\ & = \langle x, T^*(y)\rangle + \langle x, U^*(y)\rangle \end{aligned}$$

$$= \langle x, T^*(y) + U^*(y)\rangle$$
$$= \langle x, (T^* + U^*)y\rangle$$
$$\therefore \langle (T+U)x, y\rangle = \langle x, (T^* + U^*)y\rangle$$
$$\Rightarrow \langle x, (T+U)^*y\rangle = \langle x, (T^* + U^*)y\rangle$$

By the uniqueness of adjoint implies

$$(T+U)^* = T^* + U^*$$

(ii) Let $\alpha \in F$ and $x, y \in V$

$$\langle (\alpha T)(x), y\rangle = \langle \alpha T(x), y\rangle$$
$$= \alpha\langle T(x) y\rangle$$
$$= \alpha\langle x, T^*(y)\rangle$$
$$\langle (\alpha T)x, y\rangle = \langle x, \bar{\alpha}T^*(y)\rangle$$
$$\therefore \langle x, (\alpha T)^* y\rangle = \langle x, \bar{\alpha}T^*(y)\rangle$$

By the uniqueness of the adjoint implies

$$(\alpha T)^* = \bar{\alpha}T^*$$

(iii) Let $x, y \in V$

$$\langle (TU)(x), y\rangle = \langle T(U(x)), y\rangle$$
$$= \langle U(x), T^*(y)\rangle$$
$$\langle (TU)(x), y\rangle = \langle x, U^*(T^*(y))\rangle$$
$$= \langle x, (U^*T^*)(y)\rangle$$
$$\therefore \langle (TU)(x), y\rangle = \langle x, (U^*T^*)(y)\rangle$$
$$\langle x, (TU)^* y\rangle = \langle x, (U^*T^*)(y)\rangle$$

By the uniqueness the adjoint implies

$$(TU)^* = U^*T^*$$

(iv) Let $x, y \in V$

$$\langle T^*(x), y\rangle = \overline{\langle y, T^*(x)\rangle}$$
$$= \overline{\overline{\langle T(y), x\rangle}}$$
$$= \langle x, T(y)\rangle$$
$$\therefore (T^*(x), y) = \langle x, T(y)\rangle$$
$$\langle x, (T^*)^*(y)\rangle = \langle x, T(y)\rangle$$

By uniqueness of adjoint implies

$$(T^*)^* = \mathrm{T}$$

(v|) Let $x, y \in V$

$$\langle \mathrm{I}x, y\rangle = \langle x, y\rangle$$

$$= \langle x, \mathrm{I}y\rangle (\because \mathrm{I(y) = y})$$

$$\Rightarrow \langle x, \mathrm{I} * (y)\rangle = \langle x, \mathrm{I}y\rangle$$

By uniqueness of adjoint implies

$$\mathrm{I}^* = \mathrm{I}$$

3.1. INNER PRODUCT

Definition: Let V be a vector space over a ficld F, An inner product on V is a function from $V \times V \rightarrow F$ that assigns, to ever ordered pair of vectors x and y in V, a scalar in F, denoted by $\langle x, y\rangle$ such that for all $x, y, z \in V$ and scalar $\alpha \in F$ the following axioms hold:

$$I_1: \langle x, x\rangle > 0 \text{ if } x \neq 0$$
$$I_2: \langle x+z, y\rangle = \langle x, y\rangle + (z, y)$$
$$I_3: \langle \alpha x, y\rangle = \alpha(x, y)$$

I_4: $\overline{\langle x, y\rangle} = \langle y, x\rangle$, where the bar denotes the complex conjucation.

Note:

For real numbers i.e., $F = R$, the complex conjugate of a number is itself. Then I_3 reduces to

$$\langle x, y\rangle = \langle y, x\rangle$$

Properties of inner product:

If V is an inner product space, then for $x, y, z \in V$ and scalar $\alpha \in F$ the following statements are true.

(i) $\langle x, 0\rangle = (0, x) = 0$

(ii) $(x, x) = 0$ if and only if $x = 0$

(iii) $\langle x, \alpha y\rangle = \bar{\alpha}(x, y)$

(iv) $\langle x, y+z\rangle = \langle x, y\rangle + \langle x, z\rangle$

(v) $(x, y) = \langle x, z\rangle$ for all $x \in V$, then $y = z$.

Proof:

$$\begin{aligned} \text{(i) } \langle 0, x\rangle &= \langle 0+0, x\rangle \\ &= \langle 0, x\rangle + \langle 0, x\rangle = 0 \\ \therefore \langle x, 0\rangle &= \overline{\langle 0 x\rangle} = \bar{0} = 0 \end{aligned}$$

(ii) $(x, x) = 0$ if and only if $x = 0$

Let $x = 0$. Then $\langle x, x\rangle = \langle 0,0\rangle = 0$

We know that $\langle x, x\rangle > 0$ if $x \neq 0$

Obviously $\langle x, x\rangle = 0$ if and only if $x = 0$.

(iii)

$$(x, ay) = \overline{\langle ay, x\rangle}$$
$$= \overline{\bar{\alpha}(\bar{y},\bar{x})}$$
$$= \bar{\alpha}(y, x)$$
$$= \bar{\alpha}(x, y)$$

$\therefore \overline{\langle \alpha x, y\rangle} = \bar{\alpha}(x, y)$ $\quad = \overline{\langle y + z, x\rangle}$

(iv) $(x, y+z) = \overline{\langle\langle y, x\rangle} + \overline{\langle z, x\rangle} + \langle x, z\rangle$

$= \therefore (x, y+z) = \langle x, y\rangle + \langle x, z\rangle$

(iv) $(x, y+z) = \overline{(y+z, x)}$

$= \overline{\langle y, x\rangle} + \overline{\langle z, x\rangle} = \langle x, y) + (x, z) \therefore (x, y+z) = \langle x, y\rangle + (x, z)$

(v) Assume $\langle x, y\rangle = \langle x, z\rangle$ ... (1), for all $x \in V$

Consider $(x, y-z) = \langle x, y\rangle - \langle x, z\rangle$
$= \langle x, y\rangle - \langle x, y\rangle$[From (iv)]
$= 0$ ... (2)

Take $x = y - z$, we get,

$$\langle y-z, y-z\rangle = 0$$
$$\Rightarrow y - z = 0$$

$$\Rightarrow y = z$$

If $x \neq y$, then from (2), we get

Either $x = 0$ or $y - z = 0$

$\therefore y = z$

Definition: Inner product space

A vector space endowed with a specific inner product is called product space.

Standard inner product of F^n

Let $x, y \in F^n$. Then $x = (a_1, a_2, \ldots, a_n)$ and $y = (b_1, b_2, \ldots, b_n)$. The inner $\langle x, y\rangle = a_1\bar{b}_1 + a_2\bar{b}_2 + \cdots + a_n\bar{b}_n$

is called standard inner product on F^n.

Standard inner product of R^n

Let $x, y \in R^n$. Then $x = (a_1, a_2, \dots, a_n)$ and $y = (b_1, b_2, \dots, b_n)$. The inner $\langle x, y\rangle = a_1b_1 + a_2b_2 + \cdots + a_nb_n$ is called standard inner product on R^n.

3.1.1. PROBLEMS UNDER INNER PRODUCT SPACE

1. Let $x = (a_1, a_2, \dots, a_n), y = (b_1, b_2, \dots, b_n) \in F^n$. Define inner product $\langle x, y\rangle = a_1\bar{b}_1 + a_2\bar{b}_2 + \cdots + a_n\bar{b}_n$. Verify F^n is an inner space.

Sol: Let $x, y, z \in V$ and $\alpha \in F$.

Let $x = (a_1, a_2, \dots, a_n); y = (b_1, b_2, \dots, b_n)$ and $z = (c_1, c_2, \dots, c_n)$
Given $\langle x, y\rangle = a_1\bar{b}_1 + a_2\bar{b}_2 + \cdots + a_n\bar{b}_n$

I_1: $\langle x, x\rangle > 0$ if $x \neq 0$

$\langle x, x\rangle = a_1\bar{a}_1 + a_2\bar{a}_2 + \cdots + a_n\bar{a}_n$
$= |a_1|^2 + |a_2|^2 + \cdots + |a_n|^2 > 0$ [$\because a_i \neq 0$ for some i]
$\therefore \langle x, x\rangle > 0$ if $x \neq 0$

$\langle x, x\rangle = a_1\bar{a}_1 + a_2\bar{a}_2 + \cdots + a_n\bar{a}_n$

$= |a_1|^2 + |a_2|^2 + \cdots + |a_n|^2 > 0$

$\therefore (x, x) > 0$ if $x \neq 0$

I_2: $\langle x + z, y\rangle = \langle x, y\rangle + \langle z, y\rangle$

$x + z = (a_1, a_2, \dots, a_n) + (c_1, c_2, \cdots, c_n) = (a_1 + c_1, a_2 + c_2, \dots, a_n + c_n)$
$(x + z, y) = (a_1 + c_1)\bar{b}_1 + (a_2 + c_2)\bar{b}_2 + \cdots + (a_n + c_n)\bar{b}_n = a_1\bar{b}_1 + a_2\bar{b}_2 +$

$\cdots + a_n\bar{b}_n + c_1\bar{b}_1 + c_2\bar{b}_2 + \cdots + c_1^2 + x, x)$

$= a_1\bar{b}_1 + a_2\bar{b}_2 + \cdots + a_n\bar{b}_n + c_1\bar{b}_1 + c_2\bar{b}_2 + \cdots + c_n\bar{b}_n$

$= \langle x, y\rangle + \langle z, y\rangle$

$\langle x + z, y\rangle = \langle x, y\rangle + \langle z, y\rangle$

I_3: $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$

We have $x = (a_1, a_2, \dots, a_n)$.

$\therefore \alpha x = (\alpha a_1, \alpha a_2 + \cdots, \alpha a_n)(\alpha x, y) = \alpha a_1\bar{b}_1 + \alpha a_2\bar{b}_2 + \cdots + \alpha a_n\bar{b}_n$
$= \alpha(a_1\bar{b}_1 + a_2\bar{b}_2 + \cdots + a_n\bar{b}_n) = \alpha\langle x, y\rangle \therefore \langle \alpha x, y\rangle = \alpha\langle x, y\rangle$

I_4: $\overline{\langle x, y\rangle} = \langle y, x\rangle$ $\langle x, y\rangle = a_1\bar{b}_1 + a_2\bar{b}_2 + \cdots + a_n\bar{b}_n$ $\overline{\langle x, y\rangle}$

$= \overline{a_1\bar{b}_1 + a_2\bar{b}_2 + \cdots + a_n\bar{b}_n} = \bar{a}_1 b_1 + \bar{a}_2 b_2 + \cdots + \bar{a}_n b_n$

$= b_1\bar{a}_1 + b_2\bar{a}_2 + \cdots + b_n\bar{a}_n = (y, x)$ $\therefore \overline{\langle x, y\rangle} = \langle y, x\rangle$

2. Consider the vector space R^n. Prove that R^n is an inner product space with inner product $\langle x, y\rangle = a_1b_1 + a_2b_2 + \cdots + a_nb_n$ where $x = (a_1, a_2, \ldots, a_n)$ and $y = (b_1, b_2, \ldots, b_n)$.

Sol: Let $x, y, z \in V$ and $\alpha \in F$.

Let $x = (a_1, a_2, \ldots, a_n)$; $y = (b_1, b_2, \ldots, b_n)$ and $z = (c_1, c_2, \ldots, c_n)$

Given $\langle x, y\rangle = a_1b_1 + a_2b_2 + \cdots + a_nb_n$

I_1: $(x, x) > 0$ if $x \neq 0$

$\langle x, x\rangle = a_1a_1 + a_2a_2 + \cdots + a_na_n$
$= a_1^2 + a_2^2 + \cdots + a_n^2 > 0$ [$\because a_i \neq 0$ for some i]

$\therefore (x, x) > 0$ if $x \neq 0$

I_2: $(x + z, y) = \langle x, y\rangle + \langle z, y\rangle$ $x + z = (a_1, a_2, \ldots, a_n) + (c_1, c_2, \ldots, c_n)$

$= (a_1 + c_1, a_2 + c_2, \ldots, a_n + c_n)$ $(x + z, y)$

$= (a_1 + c_1)b_1 + (a_2 + c_2)b_2 + \cdots + (a_n + c_n)b_n$

$= a_1b_1 + a_2b_2 + \cdots + a_nb_n + c_1b_1 + c_2b_2 + \cdots + c_nb_n$

$= \langle x, y\rangle + \langle z, y\rangle$ $(x + z, y) = \langle x, y\rangle + (z, y)$ I_3: $\langle \alpha x, y\rangle = \alpha(x, y)$

We have $x = (a_1, a_2, \ldots, a_n)$.

$\alpha x = (\alpha a_1, \alpha a_2, \ldots, \alpha a_n)$ $\langle \alpha x, y\rangle = \alpha a_1b_1 + \alpha a_2b_2 + \cdots + \alpha a_nb_1$
$= \alpha(a_1b_1 + a_2b_2 + \cdots + a_nb_n) = \alpha\langle x, y\rangle$

$\therefore \langle \alpha x, y\rangle = \alpha\langle x, y\rangle$

I_4: $\overline{\langle x, y\rangle} = \langle y, x\rangle$

$\langle x, y\rangle = a_1b_1 + a_2b_2 + \cdots + a_nb_n$

$\overline{\langle x, y\rangle} = \overline{a_1b_1 + a_2b_2 + \cdots + a_nb_n}$
$= a_1b_1 + a_2b_2 + \cdots + a_nb_n$
$= b_1a_1 + b_2a_2 + \cdots + b_na_n$
$= \langle y, x\rangle$

$\therefore \overline{\langle x, y\rangle} = \langle y, x\rangle$

Hence R^n is an inner product space.

3. Prove that R^2 is an inmer product space with an inner product defined by $\langle x, y\rangle = a_1b_1 - a_2b_1 - a_1b_2 + 2a_2b_2$ where $x = (a_1, a_2)$; $y = (b_1, b_2)$.

Sol; Let $x, y, z \in R^2$ and $\alpha \in F$

Lat $x = (a_1, a_2); y = (b_1, b_2)$ and $z = (c_1, c_2)$

Given $\langle x, y\rangle = a_1b_1 - a_2b_1 - a_1b_2 + 2a_2b_2$

I_1 ; $\langle x, x\rangle > 0$ if $x \neq 0$

$\langle x, x\rangle = a_1a_1 - a_2a_1 - a_1a_2 + 2a_2a_2 = a_1^2 - 2a_1a_2 + 2a_2^2$

$= a_1^2 - 2a_1a_2 + a_2^2 + a_2^2$

$= (a_1 - a_2)^2 + a_2^2 > 0$ [$\because a_1 \neq 0$ or $a_2 \neq 0$]

$\therefore \langle x, x\rangle > 0$ if $x \neq 0$

I_2; $\langle x + z, y\rangle = \langle x, y\rangle + \langle z, y\rangle x + z = (a_1, a_2) + (c_1, c_2)$

$= (a_1 + c_1, a_2 + c_2)\langle x + Z, y\rangle$

$= (a_1 + c_1)b_1 - (a_2 + c_2)b_1 - (a_1 + c_1)b_2 + 2(a_2 + c_2)b_2$

$= a_1b_1 + c_1b_1 - a_2b_1 - c_2b_1 - a_1b_2 - c_1b_2 + 2a_2b_2 + 2c_2b_2$

$= a_1b_1 - a_2b_1 - a_1b_2 + 2a_2b_2 + c_1b_1 - c_2b_1 - c_1b_2 + 2c_2b_2$

$= \langle x, y\rangle + \langle z, y\rangle \therefore \langle x + z, y\rangle = \langle x, y\rangle + \langle z, y\rangle$

I_3: $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$

We have $x = (a_1, a_2)$

$\therefore \alpha x = (\alpha a_1, \alpha a_2)$

$$\langle \alpha x, y\rangle = \alpha a_1b_1 - \alpha a_2b_1 - \alpha a_1b_2 + 2\alpha a_2b_2$$
$$= \alpha(a_1b_1 - a_2b_1 - a_1b_2 + 2a_2b_2)$$
$$= \alpha\langle x, y\rangle$$

$\therefore \langle \alpha x, y\rangle = \alpha\langle x, y\rangle$

I_4: $\overline{\langle x y\rangle} = \langle y, x\rangle$

$$\langle x, y\rangle = a_1b_1 - a_2b_1 - a_1b_2 + 2a_2b_2$$
$$\overline{\langle x y\rangle} = \overline{a_1b_1 - a_2b_1 - a_1b_2 + 2a_2b_2}$$
$$= a_1b_1 - a_2b_1 - a_1b_2 + 2a_2b_2$$
$$= b_1a_1 - b_2a_2 - a_2b_1 + 2b_2a_2$$
$$= \langle y, x\rangle$$

$\therefore \overline{\langle x y\rangle} = \langle y, x\rangle$

4. Let V be the set of all real functions defined on the clo interval $[0, 1]$. The inner product on V is defined by $\langle f, g\rangle = \int_{-1}^{1} f(t)g(t)$ Prove that $V(R)$ is an inner product space.

Sol:

Let $f, g, h \in V$ and $\alpha \in F$.

Hence R^2 is an inner product space with the given inner product.

$$\text{Given } \langle f, g\rangle = \int_0^1 f(t)g(t)dt$$

$$I_1: \langle f, f\rangle > 0 \text{ if } f \neq 0$$

$$\langle f, f\rangle = \int_{-1}^{1} f(t)f(t)dt$$

$$= \int_{-1}^{1} [f(t)]^2 dt > 0$$

$$\therefore \langle f, f\rangle > 0 \text{ if } f \neq 0$$

$$I_2: \langle f + h, g\rangle = \langle f, g\rangle + \langle h, g\rangle$$

$$\langle f + h, g\rangle = \int_{-1}^{1} [f(t) + h(t)]g(t)dt$$

$$= \int_{-1}^{1} f(t)g(t))\, dt + \int_{-1}^{1} h(t)g(t)dt$$

$$= \langle f, g\rangle + \langle h, g\rangle$$

$$\therefore \langle f + h, g\rangle = \langle f, g\rangle + \langle h, g\rangle$$

$$I_3: \langle \alpha f, g\rangle = \alpha\langle f, g\rangle$$

$$\langle \alpha f, g\rangle = \int_{-1}^{1} (\alpha f)(t)g(t)dt$$

$$= \alpha \int_{-1}^{1} f(t)g(t)dt$$
$$= \alpha\langle f, g\rangle$$
$$\therefore \langle(\alpha f, g) = \alpha\langle f, g\rangle$$
$$I_4: \overline{\langle \bar{f}, \bar{g}\rangle} = \langle g, f\rangle$$
$$\langle f, g\rangle = \int_{-1}^{1} f(t)g(t)dt$$
$$\overline{\langle \bar{f}, \bar{g}\rangle} = \int_{-1}^{1} f(t)g(t)dt$$
$$= \int_{-1}^{1} f(t)g(t)dt$$
$$= \int_{-1}^{1} g(t)f(t)dt$$
$$= \langle g, f\rangle$$
$$\therefore \overline{\langle \bar{f}, \bar{g}\rangle} = \langle g, f\rangle$$

Therefore $V(R)$ is an inner product space.

5. Let H be the vector space of all continuous complex value functions on $[0, 1]$. Show that V is a complex inner product space with is product $\langle f, g\rangle = \frac{1}{3\pi}\int_0^1 f(t)\overline{g(t)}dt$.

Sol:

Let $f, g, h \in V$ and $a \in F$.

$$\text{Given } \langle f, g\rangle = \frac{1}{2\pi}\int_0^1 f(t)g(t)dt$$
$$l_1: \langle f, f\rangle > 0 \text{ if } f \neq 0$$
$$\langle f, f\rangle > 0 \text{ for } f \neq 0$$

$$\langle f, f\rangle = \frac{1}{2\pi}\int_0^1 f(t)\overline{f(t)}dt$$
$$= \frac{1}{2\pi}\int_0^1 |f(t)|^2 dt > 0$$

$\therefore (f, f) > 0$ if $f \neq 0$

$I_2 : (f+h,g) = \langle f,g\rangle + \langle h,g\rangle \langle f+h,g\rangle = \frac{1}{2\pi}\int_0^1 (f+h)(t)\overline{g(t)}dt$

$= \frac{1}{2\pi}\int_0^1 -[f(t)+h(t)]\overline{g(t)}dt$

$= \frac{1}{2\pi}\int_0^1 f(t)\overline{g(t)}dt + \frac{1}{2\pi}\int_0^1 h(t)\overline{g(t)}dt = \langle f,g\rangle + \langle h,g\rangle$

$\therefore \langle f+h,g\rangle = \langle f,g\rangle + \langle h,g\rangle I_3 : \langle \alpha f,g\rangle = \alpha\langle f,g\rangle \langle \alpha f,g\rangle$

$= \frac{1}{2\pi}\int_0^1 (\alpha f)(t)\overline{g(t)}dt = a\frac{1}{2\pi}\int_0^1 f(t)\overline{g(t)}dt = \alpha\langle f,g\rangle$

$\therefore \langle \alpha f,g\rangle = \alpha\langle f,g\rangle I_4 : \overline{(f,g)} = (g,f)(f,g)$

$= \frac{1}{2\pi}\int_0^1 f(t)\overline{g(t)}dt (f,g) = \frac{1}{2\pi}\int_0^1 f(t)\overline{g(t)}dt$

Therefore $V(C)$ is an inner product space.

3.1.2. NORM OF A VECTOR

Definition

Let V be an inner product space and let $x \in V$ then norm or length of x is $\| x \|$ and is defined by $\| x \| = \sqrt{\langle x,x\rangle}$

9. Find the norm of the following vectors in $V_3(R)$ with, inner product: $(a)(1,1,1)$, $(b)(1,2,3)$, $(c)(3,-4,0)$, $(d)(4x+5y)$ where $x = (1,-1,0)$ and $y = (1,2,3)$

Sol:

Let x=(a_1,a_2,a_3) ; y=$(b_1,b_2,b_3) \in V_3$ (R)

$$Thestandardinnerproductspaceis$$

$$\langle x,y\rangle = \langle x,y\rangle = a_1b_1 + a_2b_2 + a_3b_3$$
$$\therefore (x,x) = a_1^2 + a_2^2 + a_3^3$$

(a)Let $x = (1,1,1)$

$$\| x \|^2 = \langle x, x \rangle = 1^2 + 1^2 + 1^2 = 3$$
$$\Rightarrow \| x \| = \sqrt{3}$$

(b) Let $x = (1,2,3)$

$$\| x \|^2 = \langle x, x \rangle = 1^2 + 2^2 + 3^2 = 14$$
$$\Rightarrow \| x \| = \sqrt{14}$$

(c) Let $x = (3, -4, 0)$

$$\| x \|^2 = 3^2 + (-4)^2 + 0^2 = 9 + 16 = 25$$
$$\Rightarrow \| x \| = 5$$

(d) Let $u = 4x + 5y$

$$= 4(1,-1,0) + 5(1,2,3)$$
$$= (4,-4,0) + (5,10,15)$$
$$= (9,6,15)$$
$$\| u \|^2 = \langle u, u \rangle = 9^2 + 6^2 + 15^2 = 342$$
$$\Rightarrow \| u \| = \sqrt{342}$$

10. Find the norm of the following vectors in Euclidean space R^3 with standard inner product (a) $u = (2, 1, -1)$, (b) $v = (\frac{1}{2}, \frac{2}{3}, -\frac{1}{4})$

Sol:

(a) Let $u = (2,1,-1)$

$\| u \|^2 = 2^2 + 1^2 + (-1)^2$

$=6$

$\| u \| = \sqrt{6}$

(b) Let $v = (\frac{1}{2}, \frac{2}{3}, -\frac{1}{4})$

$\| v \|^2 = 6^2 + 8^2 + (-3)^2$

=109

$\| v \| = \sqrt{109}$

11. Find the norm of the following vectors in F^3 with standard inner product: $x = (1 + i, 2, i), y = (3i, 2 + 3i, 4)$. Find (a) $\| x \|$, (b) $\| y \|$, (c) $\| x + y \|$, (d) $\langle x, y \rangle$

Sol: Let $x, y, z \in F^3$

Let $x = (a_1, a_2, a_3); y = (b_1, b_2, b_3)$

$\langle x, y \rangle = a_1 \bar{b}_1 + a_2 \bar{b}_2 + a_3 \bar{b}_3$

$\langle x, x \rangle = |a_1|^2 + |a_2|^2 + |a_3|^2$

Then

Sol:

(a) $\| x \|^2 = \langle x, x \rangle$

$= |1 + i|^2 + |2|^2 + |i|^2$

$= 1^2 + 1^2 + 2^2 + 1^2$

$= 7$

$\| x \| = \sqrt{7}$

(b) $\| y \|^2 = \langle y, y \rangle$

$= |3i|^2 + |2 + 3i|^2 + |4|^2$

$= 3^2 + 2^2 + 3^2 + 4^2$

$= 9 + 4 + 9 + 16$

$= 38$

$\| y \| = \sqrt{38}$

(c) $x + y = (1 + i, 2, i) + (3i, 2 + 3i, 4)$

$= (1 + 4i, 4 + 3i, 4 + i)$

$\| x + y \|^2 = |1 + 4i|^2 + |4 + 3i|^2 + |4 + i|^2$

$= 1^2 + 4^2 + 4^2 + 3^2 + 4^2 + 1^2$

$= 59$

$\| x + y \| = \sqrt{59}$

(d) $\langle x, y \rangle = \langle (1 + i, 2, i), (3i, 2 + 3i, 4) \rangle$

$= (1 + i)(\bar{3i}) + 2(2 + 3i) + i4$

$= (1 + i)(-3i) + 2(2 - 3i) + 4i$

$= -3i + 3 + 4 - 6i + 4i$

$= 7 - 5i$

12. Let V be an vector space of polynomials with the inner product given by $\langle f, g \rangle = \int_0^1 f(t) g(t) dt$. Let $f(t) = t + 2$ and $g(t) = t^2 - 2t - 3$ find (i)

$\langle \boldsymbol{f}, \boldsymbol{g} \rangle$ **(ii)** $\| \boldsymbol{f} \|$.

Sol:

$$\text{Let}\langle f, g\rangle = \int_0^1 f(t)g(t)dt$$

(i)

$$= \int_0^1 (t+2)(t^2-2t-3)dt$$

$$= \int_0^1 (t^3-2t^2-3t+2t^2-4t-6)dt$$

$$= \int_0^1 (t^3-7t-6)dt$$

$$= \left[\frac{t^4}{4} - \frac{7t^2}{2} - 6t\right]_0^1$$

$$= \frac{1}{4} - \frac{7}{2} - 6$$

$$= -\frac{37}{4}$$

ii)

$$\| f \|^2 = \langle f, f\rangle$$

$$= \int_0^1 [f(t)]^2 dt$$

$$= \int_0^1 (t+2)^2 dt$$

$$= \int_0^1 (t^2+4t+4)dt$$

$$= \left[\frac{t^3}{3} + \frac{4t^2}{2} + 4t\right]_0^1$$

$$= \frac{1}{3} + 2 + 4$$

$$= \frac{19}{3}$$

$$\| f \| = \frac{\sqrt{19}}{\sqrt{3}}$$

13. For any non-zero vector, $x \in V$. prove that $y = \frac{x}{\|x\|}$ is a vector such that $\| y \| = 1$.

Sol: Consider

$$\begin{aligned}\langle y, y\rangle &= \left\langle \frac{x}{\| x \|}, \frac{x}{\| x \|}\right\rangle \\ &= \frac{1}{\| x \|} \cdot \frac{1}{\| x \|}\langle x, x\rangle \\ \langle y, y\rangle &= \frac{1}{\| x \|^2} \| x \|^2 \\ \| y \|^2 &= 1 \\ \| y \| &= 1\end{aligned}$$

Theorem 3.1: In an inner product space V,

(i) $\| x \| \geq 0$ and $\| x \| = 0$ if and only if $x = 0$

(ii) $\| \alpha x \| = |\alpha||x|$

Proof:

(i)

$$\| x \| = \sqrt{\langle x, x\rangle}$$
$$\| x \|^2 = \langle x, x\rangle \geq 0$$
$$\| x \|^2 \geq 0$$
$$\| x \| \geq 0$$

Also $(x, x) \geq 0$ if and only if $x = 0$

Therefore $\| x \|^2 = 0$ if and only if $x = 0$

(ii)

$$\begin{aligned}\| \alpha x \|^2 &= \langle \alpha x, \alpha x\rangle \\ &= \alpha\langle x, \alpha x\rangle \\ &= \alpha\bar{\alpha}\langle x, x\rangle \\ &= |\alpha|^2 \| x \|^2 \\ \| \alpha x \| &= |\alpha| \| x \|\end{aligned}$$

Theorem 3.2: [Schwarz's inequality]

For any two vectors x and y in an inner product space V,

$$|\langle \boldsymbol{x}, \boldsymbol{y}\rangle| \leq \| \boldsymbol{x} \| \| \boldsymbol{y} \|$$

Proof:

If $x = 0$, then $\| x \| = 0$.
$\therefore \| x \| \| y \| = 0 \ldots$ (1)
Also $\langle x, y \rangle = \langle 0, y \rangle = 0$
$\therefore |\langle x, y \rangle| = 0 \ldots$ (2).

From (1) and (2)

$$|\langle x, y \rangle| = \| x \| \| y \|$$

So the result is true.

Let $x \neq 0$. Then $\| x \| > 0$

Therefore $\frac{1}{\|x\|}$ is a positive number

Consider the vector

$$w = y - \frac{\langle y, x \rangle}{\| x \|^2} x$$

$$\langle w, w \rangle = \langle y - \frac{\langle y, x \rangle}{\| x \|^2} x, y - \frac{\langle y, x \rangle}{\| x \|^2} x \rangle$$

$$= \langle y, y \rangle - \langle y \left[\frac{(y, x)}{\| x \|^2} ; \right\rangle - \langle \frac{\langle y, x \rangle}{\| x \|^2} x, y \rangle + \langle \frac{\langle y, x \rangle}{\| x \|^2} x, \frac{\langle y, x \rangle}{\| x \|^2} x \rangle$$

$$= \| y \|^2 - \frac{\text{Inner Product}}{\|x\|^2} \langle y, x \rangle - \frac{\langle y,x \rangle}{\|x\|^2} \langle x, y \rangle + \frac{\langle y,x \rangle \langle y,x \rangle}{\|x\|^4} \langle x, x \rangle$$

$$= \| y \|^2 - \frac{\langle x,y \rangle \langle \langle x,y \rangle}{\|x\|^2} - \frac{\frac{\langle x,y \rangle}{\langle x,y \rangle}}{\|x\|^2} + \frac{\overline{\langle x,y \rangle} \langle x,y \rangle \|x\|^2}{\|x\|^4}$$

$$= \| y \|^2 - \frac{|\langle x, y \rangle|^2}{\| x \|^2} - \frac{|\langle x, y \rangle|^2}{\| x \|^2} + \frac{|\langle x, y \rangle|^2}{\| x \|^2} \left[\because z\bar{z} = |z|^2\right]$$

$$(w, w) = \| y \|^2 - \frac{|\langle x, y \rangle|^2}{\| x \|^2}$$

$$\therefore \| y \|^2 - \frac{|\langle x, y \rangle|^2}{\| x \|^2} \geq 0$$

$$\| x \|^2 \| y \|^2 - |\langle x, y \rangle|^2 \geq 0$$

$\| x \|^2 \| y \|^2 \geq |\langle x, y \rangle|^2$ $|\langle x, y \rangle|^2 \leq \| x \|^2 \| y \|^2$

$: |\langle x, y \rangle| \leq \| x \| \| y \|$

Theorem 3.3: [Triangle inequality]

For any two vectors x and y in an inner product space V,

$$\| x + y \| \leq \| x \| + \| y \| .$$

Proof:

Using the norm of vectors we have

$$\begin{aligned}
| x + y \|^2 &= \langle x + y, x + y\rangle \\
&= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle \\
&= \| x \|^2 + \langle x, y\rangle + \overline{\langle x, y\rangle} + \| y \|^2 \\
&= \| x \|^2 + 2\mathrm{Re}\langle x, y\rangle + \| y \|^2 \; [\because z + \bar{z} = 2\mathrm{Re}(z)] \\
&\leq \| x \|^2 + 2|\langle x, y\rangle| + \| y \|^2 \; [\because \mathrm{Re}(z) \leq |z|] \\
&\leq \| x \|^2 + 2 \| x \| \| y \| + \| y \|^2 \; [\text{ By Shwarz's inequivality }] \\
&\leq (\| x \| + \| y \|)^2 \\
| x + y \|^2 &\leq (\| x \| + \| y \|)^2 \\
\| x + y \| &\leq \| x \| + \| y \|.
\end{aligned}$$

Theorem 3.4 : [Parallelogram law]

For any two vectors x and y in an inner product space V,

$\| x + y \|^2 + \| x - y \|^2 = 2(\| x \|^2 + \| y \|^2)$. What does this equation state about parallelograms in R^2 ?

Proof:

$$\begin{aligned}
\| x + y \|^2 &= \langle x + y, x + y\rangle \\
&= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle \\
&= \| x \|^2 + \langle x, y\rangle + \bar{x}, y + \| y \|^2 \\
&= \| x \|^2 + 2\mathrm{Re}\langle x, y\rangle + \| y \|^2 \; \ldots (1)
\end{aligned}$$

and

$$\begin{aligned}
\| x - y \|^2 &= \langle x - y, x - y\rangle \\
&= \langle x, x\rangle - \langle x, y\rangle - \langle y, x\rangle + \langle y, y\rangle \\
&= \| x \|^2 - [\langle x, y\rangle + \langle x, y\rangle] + \| y \|^2 \\
&= \| x \|^2 - 2\mathrm{Re}\langle x, y\rangle + \| y \|^2 \; \ldots (2)
\end{aligned}$$

$(1) + (2) \Rightarrow$

$$\| x + y \|^2 + \| x - y \|^2 = 2(\| x \|^2 + 2 \| y \|^2) \ldots (3)$$

Let $0ABC$ be a parallelogram with sides of length $0A = \| x \|$ and $0C = \| y \|$ in R^2. Therefore the length of the hypotenuses of $0ABC$ are $AC = \| x + y \|$ and $OB = \| x - y \|$

(3) $\Rightarrow 0B^2 + AC^2 = 0A^2 + AB^2 + BC^2 + CA^2 [\because |0A| = |BC|, |AB| = |C0|]$

Therefore sum of the squares of the two diagonals is equal to the sum of squares of four sides.

3.3.1 PROBLEMS UNDER LEAST SQUARES TO FIT A STRAIGFII LINE

To find the least square fit of $y = ct + d$ for the n datas $(t_1, y_1), (t_2, y_2), \ldots, (t_n, y_n)$, the appropriate model is

$$A = \begin{pmatrix} t_1 & 1 \\ t_2 & 1 \\ \vdots & \vdots \\ t_n & 1 \end{pmatrix}, x_0 = \begin{pmatrix} c \\ d \end{pmatrix}, \text{ and } y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

To compute x_0, the normal equations are

$$(A^*A)x_0 = A^*y.$$

By least square find a linear function and error for the following data $(1, 0), (2, 1)$, and $(3, 3)$

Sol: Let $y = ct + d$ be the best fit.

Here $t_1 = 1, t_2 = 2, t_3 = 3$

$y_1 = 0, y_2 = 1, y_3 = 3$

$$A = \begin{bmatrix} t_1 & 1 \\ t_2 & 1 \\ t_3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}; x_0 = \begin{bmatrix} c \\ d \end{bmatrix}; y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$$

$$A^* = (\bar{A})^T = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

$$A^*A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 14 & 6 \\ 6 & 3 \end{bmatrix}$$

$$A^*y = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 11 \\ 4 \end{bmatrix}$$

Normal equals are

$$(A^*A)x_0 = A^*y$$
$$\begin{bmatrix} 14 & 6 \\ 6 & 3 \end{bmatrix}\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 11 \\ 4 \end{bmatrix}$$
$$14c + 6d = 11 \ldots (1)$$
$$6c + 3d = 4 \ldots (2)$$

Solve (1) and (2)

$(1) \times 3 \Rightarrow 42c + 18d = 33$

$(2) \times 7 \Rightarrow 42c + 21d = 28$

Subtracting

$$-3d = 5$$
$$\therefore d = \frac{-5}{3}$$

Substituting $d = \frac{-5}{3}$ in (2),

$$6c + 3\left(\frac{-5}{3}\right) = 4$$
$$6c = 9$$
$$c = \frac{3}{2}$$

$\therefore y = \frac{3}{2}t - \frac{5}{3}$ least square fit.

The error is computed using the formula

$$E = \|Ax_0 - y\|^2$$

where

.

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix};\ x_0 = \begin{bmatrix} \frac{3}{2} \\ -\frac{5}{3} \end{bmatrix};\ y = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$$

$$Ax_0 - y = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} \frac{3}{2} \\ -\frac{5}{3} \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{6} \\ \frac{4}{3} \\ \frac{17}{6} \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{6} \\ \frac{1}{3} \\ -\frac{1}{6} \end{bmatrix}$$

$\varepsilon = \|Ax_0 - y\|^2$

$= \langle Ax_0 - y, Ax_0 - y \rangle$

$= (-\frac{1}{6})^2 + (-\frac{1}{3})^2 + (-\frac{1}{6})^2$

$= 0.0278 + 0.1111 + 0.0278$

$= 0.1667$

Find the least square line and error for the following datas (1,2), (2, 3), (3, 5) and (4, 7).

Sol: Let $y = ct + d$ be the best fit.

Here $t_1 = 1, t_2 = 2, t_3 = 3, t_4 = 4$

$y_1 = 2, y_2 = 3, y_3 = 5, y_4 = 7$

$A = \begin{bmatrix} t_1 & 1 \\ t_2 & 1 \\ t_3 & 1 \\ t_4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{bmatrix}; x_0 = \begin{bmatrix} c \\ d \end{bmatrix}; y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 5 \\ 7 \end{bmatrix} A^* = (\bar{A})^T$

$$= \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{bmatrix} A^*A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{bmatrix}$$

The normal equals are

$$(A^*A)x_0 = A^*B$$

$$\begin{bmatrix} 30 & 10 \\ 10 & 4 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 51 \\ 17 \end{bmatrix}$$

$$30c + 10d = 51 \ldots (1)$$

$$10c + 4d = 17 \ldots (2)$$

Solve (1) and (2)

$(1) \Rightarrow \ 30c + 10d = 51$

$(2) \times 3 \Rightarrow$

$$30c + 12d = 51$$
$$-d = 0$$
$$\therefore d = 0$$

Subtracting $-d = 0$

Substituting $d = 0$ in (2),$10c + 0 = 17$

$$10c = 17$$
$$c = 1.7$$

$\therefore y = 1.7t$ least square fit.

The error is computed using the formula

$$E = \|Ax_0 - y\|^2$$

where

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{bmatrix}; x_0 = \begin{bmatrix} 1.7 \\ 0 \end{bmatrix}; y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 5 \\ 7 \end{bmatrix}$$

$$\mathrm{I}\; x_0 - y = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1.7 \\ 0 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \\ 5 \\ 7 \end{bmatrix}$$

$$= \begin{bmatrix} 1.7 \\ 3.4 \\ 5.1 \\ 6.8 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \\ 5 \\ 7 \end{bmatrix} = \begin{bmatrix} -0.3 \\ 0.4 \\ 0.1 \\ 0.2 \end{bmatrix}$$

$$\begin{aligned} E &= \|Ax_0 - y\|^2 \\ &= (Ax_0 - y, Ax_0 - y) \\ &= (-0.3)^2 + (0.4)^2 + (0.1)^2 + (0.2)^2 \\ &= 0.09 + 0.16 + 0.01 + 0.04 \\ &= 0.3 \end{aligned}$$

3.3.2. PROBLEMS UNDER LEAST SQUARES TO FIT A QUADRATIC FUNCTION

To find the least square fit of $y = ct^2 + dt + e$ for the

$(t_1, y_1), (t_2, y_2), \ldots, (t_n, y_n)$, the appropriate model is

$$A = \begin{bmatrix} t_1^2 & t_1 & 1 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ t_n^2 & t_n & 1 \end{bmatrix};\ x_0 = \begin{pmatrix} c \\ d \\ e \end{pmatrix},\ y = \begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{bmatrix}$$

To compute x_0, the normal equations are

$(A^*A)x_0 = A^*y.$

Using the least squares fit a quadratic following data $(-3, 9), (-2, 6), (0, 2)$ and $(1, 1)$. Also find the error.

Sol: Let $y = ct^2 + dt + e$ be the best fit.

Here $t_1 = -3, t_2 = -2, t_3 = 0, t_4 = 1$ $A = \begin{bmatrix} t_1^2 & t_1 & 1 \\ t_2^2 & t_2 & 1 \\ t_3^2 & t_3 & 1 \\ t_4^2 & t_4 & 1 \end{bmatrix} = \begin{bmatrix} 9 & -3 & 1 \\ 4 & -2 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$; $x_0 = \begin{bmatrix} c \\ d \\ e \end{bmatrix}$; $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 2 \\ 1 \end{bmatrix}$

$$A^* = (\bar{A})^T$$

$$= \begin{bmatrix} 9 & 4 & 0 & 1 \\ -3 & -2 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$A^*A = \begin{bmatrix} 9 & 4 & 0 & 1 \\ -3 & -2 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 9 & -3 & 1 \\ 4 & -2 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$A^*y = \begin{bmatrix} 98 & -34 & 14 \\ -34 & 14 & -4 \\ 14 & -4 & 4 \end{bmatrix} = \begin{bmatrix} 9 & 4 & 0 & 1 \\ -3 & -2 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 9 \\ 6 \\ 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 125 \\ -40 \\ 18 \end{bmatrix}$$

Normal equals are

$$(A^*A)x_0 = A^*y$$

$$\begin{bmatrix} 98 & -34 & 14 \\ -34 & 14 & -4 \\ 14 & -4 & 4 \end{bmatrix} \begin{bmatrix} c \\ d \\ e \end{bmatrix} = \begin{bmatrix} 125 \\ -40 \\ 18 \end{bmatrix}$$

$$98c - 34d + 14e = 125 \ldots (1)$$

$$-34c + 14d - 4e = -40 \ldots (2)$$

$$14c - 4d + 4e = 18 \ldots (2)$$

Solve (1), (2) and (3), we get

$$c = \frac{1}{3}, d = -\frac{4}{3}, e = 2$$

$$y = \frac{1}{3}t^2 - \frac{4}{3}t + 2 \text{ least square fit.}$$

The error is computed using the formula

E= $\|Ax_0 - y\|^2$

where

$$A = \begin{bmatrix} 9 & -3 & 1 \\ 4 & -2 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}; x_0 = \begin{bmatrix} \frac{1}{3} \\ -\frac{4}{3} \\ 2 \end{bmatrix}; y = \begin{bmatrix} 9 \\ 6 \\ 2 \\ 1 \end{bmatrix}$$

$$Ax_0 - y = \begin{bmatrix} 9 & -3 & 1 \\ 4 & -2 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{3} \\ -\frac{4}{3} \\ 2 \end{bmatrix} - \begin{bmatrix} 9 \\ 6 \\ 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 \\ 6 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 9 \\ 6 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$E = \|Ax_0 - y\|^2 = 0$$

Using the least squares fit a quadratic function following data $(-2, 4), (-1, 3), (0, 1), (1, -1)$ and $(2, -3)$.

Sol: Let $y = ct^2 + dt + e$ be the best fit.

Here $t_1 = -2, t_2 = -1, t_3 = 0, t_4 = 1, t_5 = 2$

$y_1 = 4, y_2 = 3, y_3 = 1, y_4 = -1, y_5 = -3$

$$A = \begin{bmatrix} t_1^2 & t_1 & 1 \\ t_2^2 & t_2 & 1 \\ t_3^2 & t_3 & 1 \\ t_4^2 & t_4 & 1 \\ t_5^2 & t_5 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -2 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix}; x_0 = \begin{bmatrix} c \\ d \\ e \end{bmatrix}; y = \begin{bmatrix} 4 \\ 3 \\ 1 \\ -1 \\ -3 \end{bmatrix}$$

$$A^* = (\bar{A})^T$$

$$= \begin{bmatrix} 4 & 1 & 0 & 1 & 4 \\ -2 & -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$A^*A = \begin{bmatrix} 4 & 1 & 0 & 1 & 4 \\ -2 & -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & -2 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 34 & 0 & 10 \\ 0 & 10 & 0 \\ 10 & 0 & 5 \end{bmatrix}$$

$$A^*y = \begin{bmatrix} 4 & 1 & 0 & 1 & 4 \\ -2 & -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 1 \\ -1 \\ -3 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ -18 \\ 4 \end{bmatrix}$$

Normal equals are

$$(A^*A)x_0 = A^*y$$

$$\begin{bmatrix} 34 & 0 & 10 \\ 0 & 10 & 0 \\ 10 & 0 & 5 \end{bmatrix} \begin{bmatrix} c \\ d \\ e \end{bmatrix} = \begin{bmatrix} 6 \\ -18 \\ 4 \end{bmatrix}$$

$$34c + 10e = 6 \ldots (1)$$

$$10d = -18 \ldots (2)$$

$$10c + 5e = 4 \ldots (3)$$

$(2) \Rightarrow d = -1.8$

Solving (1) and (3), we get

$c = -0.14, e = 1.08,$

$y = -0.14t^2 - 1.8t + 1.08$ least square fit.