Catalog

Definite and indefinite Integrals

Definite Integral

The integral which has definite value is called Definite Integral. In other words, when $\int g(x)dx = f(x) + C$, then $[f(b) - f(a)]$ is called the Definite Integral of $g(x)$ between the limits (or end values) a and b and denoted by the symbol $\int_a^b g(x)dx$, a is called the lower limit and b is called the upper limit and is denoted by $[f(x)]_a^b$

Thus $\int_a^b g(x)dx = [f(x)]_a^b = [f(b) - f(a)]$

Theorem 1: If f is continuous on $[a, b]$, (or) if f has only a finite number of discontinuous, then f is integrable on $[a, b]$

i.e., $\int_a^b f(x)dx$ exists.

Theorem 2: If f is integrable on $[a, b]$ then $\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$

$$\Delta x = \frac{b-a}{n} \text{ and } x_i = a + i\Delta x$$

Example :

Evaluate $\int_0^3 (x^2 - 2x)\, dx$ by using Riemann sum by taking right end points as the sample points.

Solution:

Take n subintervals, we have $\Delta x = \frac{b-a}{n} = \frac{3}{n}$

$$x_0 = 0, x_1 = \frac{3}{n}, x_2 = \frac{3}{n}, x_3 = \frac{3}{n}, \dots, x_i = \frac{3i}{n}$$

Since we are using right end points.

$$\therefore \int_0^3 (x^2 - 2x)\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{3i}{n}\right)\left(\frac{3}{n}\right)$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left[\left(\frac{3i}{n}\right)^2 - 2\left(\frac{3i}{n}\right)\right] = \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left[\frac{9}{n^2} i^2 - \frac{6}{n} i\right]$$

$$= \lim_{n\to\infty} \frac{27}{n^3} \sum_{i=1}^{n} i^2 - \lim_{n\to\infty} \frac{18}{n^2} \sum_{i=1}^{n} i$$

$$= \lim_{n\to\infty} \frac{27}{n^3} \left[\frac{n(n+1)(2n+1)}{6}\right] - \lim_{n\to\infty} \frac{18}{n^2} \frac{n(n+1)}{2}$$

$$= \lim_{n\to\infty} \frac{27}{6n^3} n^3 \left[1 + \frac{1}{n}\right]\left[2 + \frac{1}{n}\right] - \lim_{n\to\infty} \frac{9}{n^2} n^2 \left[1 + \frac{1}{n}\right]$$

$$= \left(\frac{27}{6}\right)(1)(2) - 9 = 9 - 9 = 0$$

Example:

Evaluate the Riemann sum for f(x) = $x^3 - 6x$, taking the sample points to be right end points and a = 0, b = 3 and n= 6

Solution:

$$\Delta x = \frac{b-a}{n} = \frac{3-0}{6} = \frac{1}{2}$$

The right end points are 0.5, 1, 1.5, 2, 2.5 and 3

The Riemann sum is

$$R_6 = \sum_{i=1}^{6} f(x_i)\Delta x = \sum_{i=1}^{6} f(x_i)\left(\frac{1}{2}\right) = \frac{1}{2}\sum_{i=1}^{6} f(x_i)$$

$$= \frac{1}{2}[f(0.5) + f(1) + f(1.5) + f(2) + f(2.5) + f(3)]$$

$$= \frac{1}{2}[-2.875 - 5 - 5.625 - 4 + 0.625 + 9] = -3.9375$$

Example:

Use the definition of area to find an expression for the area under the curve of $f(x) = e^{-x}$ between $x = 0$, $x = 2$. Do not evaluate the limit.

Solution:

Given that $f(x) = e^{-x}, a = 0, b = 2$

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n}$$

$$x_i = a + i\Delta x = 0 + i\left(\frac{2}{n}\right)$$

Area under the curve $f(x) = e^{-x}$ between $x = 0$ and $x = 2$ is given by

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{2i}{n}\right)\left(\frac{2}{n}\right)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(e^{-2i/n}\right)\left(\frac{2}{n}\right)$$

The Mid Point

The Riemann sum which is the approximation to a given integral using the midpoint is given by

$$\int_a^b f(x)dx \simeq \sum_{i=1}^{n} f(\bar{x}_i)\Delta x$$

$$= \Delta x[f(\bar{x}_1) + \cdots + f(\bar{x}_n)]$$

Where $\Delta x = \frac{b-a}{n}$ and $\bar{x}_i = \frac{1}{2}[x_{i-1} + x_i]$

= midpoint of $[x_{i-1}, x_i]$

The Fundamental theorem of Calculus

Part 1: If f is continuous on $[a, b]$ then the function g is defined by

$$g(x) = \int_a^x f(t)dt\,;\ \ a \le x \le b$$

is continuous on $[a, b]$ and differentiable on $(a\,, b)$, $and\ g'(x) = f(x)$

The Fundamental theorem of Calculus

Part 2: If f is continuous on $[a, b]$ then $\int_a^b f(x)dx = F(b) - F(a)$

Where F is any anti derivative of f, that is, a function such that $F' = f$

Example :

Find the derivative of the following

(i) $g(x) = \int_0^x (t^2 + 1)\, dt$

Solution:

Given $g(x) = \int_0^x (t^2 + 1)\, dt$

$\therefore\ g'(x) = (x^2 + 1)$ $\quad(\because f(t) = t^2 + 1$ is continuous by FTC1)

(ii) $h(x) = \int_1^{e^x} \log t\, dt$

Solution:

Given $h(x) = \int_1^{e^x} \log t\, dt$

Put $u = e^x \Rightarrow du = e^x\, dx \Rightarrow \frac{du}{dx} = e^x$

$$\frac{dh}{dx} = \frac{dh}{du}\frac{du}{dx}$$

$$= \frac{d}{du}\left[\int_1^u \log t\, dt\right] e^x = \log u\,(e^x) = \log(e^x)\, e^x = xe^x$$

(iii) $f(x) = \int_0^{\tan x} \sqrt{t + \sqrt{t}}\, dt$

Solution:

Given $f(x) = \int_0^{\tan x} \sqrt{t + \sqrt{t}}\, dt$

Put $u = tanx \Rightarrow du = sec^2x\, dx \Rightarrow \frac{du}{dx} = sec^2x$

$$\frac{df}{dx} = \frac{df}{du}\frac{du}{dx}$$

$$= \frac{d}{du}\left[\int_0^u \sqrt{t + \sqrt{t}}\, dt\right] sec^2x = \sqrt{u + \sqrt{u}}\, sec^2 x =$$

$$\sqrt{tan\, x + \sqrt{tan\, x}}\, sec^2x$$

Example :

Evaluate $\int_3^6 \frac{1}{x} dx$ by fundamental theorem of calculus

Solution:

The function $f(x) = \frac{1}{x}$ is continuous in $3 \le x \le 6$.

By fundamental theorem of calculus part II, Anti derivative $F(x) = \log x$

$$\int_3^6 \frac{1}{x} dx = [log x]_3^6 = log 6 - log 3$$

$$= log\left(\frac{6}{3}\right) = log 2$$

Example:

Find the derivative of the following

(i) $\int_{-1}^{2} (x^3 - 2x) dx$

Solution:

Given $f(x) = x^3 - 2x$ is continuous in $-1 \le x \le 2$

By FTC 2, Anti derivative $F(x) = \frac{x^4}{4} - \frac{2x^2}{2} = \frac{x^4}{4} - x^2$

$\int_{-1}^{2} (x^3 - 2x) dx = F(b) - F(a) = F(2) - F(-1)$

$$= \left[\frac{2^4}{4} - 2^2\right] - \left[\frac{(-1)^4}{4} - (-1)^2\right] = \frac{3}{4}$$

(ii) $\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{8}{1+x^2} dx$

Solution:

Given $f(x) = \frac{8}{1+x^2}$ is continuous in the given interval.

By FTC 2, Anti derivative $F(x) = 8 tan^{-1} x$

$$\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{8}{1+x^2} dx = F(b) - F(a) = F(\sqrt{3}) - F\left(\frac{1}{\sqrt{3}}\right)$$

$$= 8\, tan^{-1}(\sqrt{3}) - 8\, tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$= 8\left(\frac{\pi}{3}\right) - 8\left(\frac{\pi}{6}\right) = \frac{4}{3}\pi$$

(iii) $\int_1^9 \frac{x-1}{\sqrt{x}} dx$

Solution:

Given $f(x) = \frac{x-1}{\sqrt{x}} = \sqrt{x} - \frac{1}{\sqrt{x}} = x^{1/2} - x^{-1/2}$ is continuous in the given interval.

By FTC 2, Anti derivative $F(x) = \frac{x^{3/2}}{3/2} - \frac{x^{1/2}}{1/2} = \frac{2}{3} x^{3/2} - 2\, x^{1/2}$

$$\int_1^9 \frac{x-1}{\sqrt{x}}\,dx = F(b) - F(a) = F(9) - F(1)$$

$$= \left[\frac{2}{3}(9)^{3/2} - 2(9)^{1/2}\right] - \left[\frac{2}{3} - 2\right]$$

$$= (18-6) - \left(-\frac{4}{3}\right) = 12 + \frac{4}{3} = \frac{40}{3}$$

Example:

What is wrong with the calculation $\int_0^{\pi} sec^2 x\; dx = 0$

Solution:

Given $f(x) = sec^2 x = \frac{1}{cos^2 x}$ $0 \le x \le \pi$

The fundamental theorem of calculus applies to continuous function.

Here, $f(x) = sec^2 x = \frac{1}{cos^2 x}$ is not continuous at $x = \frac{\pi}{2}$.

Since $f\left(\frac{\pi}{2}\right) = \frac{1}{cos^2 \frac{\pi}{2}} = \frac{1}{0} = \infty$

At $x = \frac{\pi}{2}$ the function $f(x) = sec^2 x$ is discontinuous.

So $\int_0^{\pi} sec^2 x\; dx$ does not exist.

Example:

What is wrong with the calculation $\int_{-1}^{3} \frac{dx}{x^2} = -\frac{4}{3}$

Solution:

The fundamental theorem of calculus applies to continuous function.

Here, $f(x) = \frac{1}{x^2}$ is not continuous at $[-1, 3]$.

That is f(x) is discontinuous at $x = 0$. So $\int_{-1}^{3} \frac{dx}{x^2}$ does not exist.

Example:

What is wrong with the calculation $\int_{\pi/3}^{\pi} sec\,\theta\; tan\,\theta\; d\theta = -3$

Solution:

Given $\int_{\pi/3}^{\pi} sec\,\theta\; tan\,\theta\; d\theta$

$$\int_{\pi/3}^{\pi} sec\,\theta\; tan\,\theta\; d\theta = [sec\,\theta]_{\pi/3}^{\pi} = -3$$

The fundamental theorem of calculus applies to continuous function.

Here, $f(\theta) = \sec\theta \tan\theta$ is not continuous on the interval $[\frac{\pi}{3}, \pi]$, since $\tan\frac{\pi}{2} = \infty$

Indefinite Integral

$\int g(x)dx = f(x) + C$ where C is the arbitrary constant of integration. By taking different values C we get any number of solution. Therefore $f(x) + C$ is called the indefinite integral of $g(x)$.

For convenience, we normally omit C when we evaluate an indefinite integral.
As the fundamental theorem of calculus establish a connection between anti derivative and integrals. Thus $\int g(x)dx = f(x)$ *means* $f'(x) = g(x)$.

Formulae

1. $\int k\, dx = kx + C$
2. $\int e^x\, dx = e^x + C$
3. $\int x^n\, dx = \frac{x^{n+1}}{n+1} + C\ (n \neq 1)$
4. $\int \frac{dx}{x} = \log x + C$
5. $\int a^x\, dx = a^x \log a + C$
6. $\int sinx\, dx = -\cos x + C$
7. $\int cosx\, dx = \sin x + C$
8. $\int sec^2 x\, dx = tanx + C$
9. $\int cosec^2 x\, dx = -cotx + C$
10. $\int sec\, x\, tan\, x\, dx = \sec x + C$
11. $\int cosec\, x\, cot\, x\, dx = -\operatorname{cosec} x + C$
12. $\int \tan x\, dx = \log \sec x + C$
13. $\int \cot x\, dx = \log \sin x + C$
14. $\int \sec x\, dx = \log(\sec x + \tan x) + C$
15. $\int cosec\, x\, dx = \log(\, cosec\, x - \cot x\,) + C$
16. $\int \frac{dx}{1+x^2} = tan^{-1} x + C$
17. $\int \frac{dx}{\sqrt{1-x^2}} = sin^{-1} x + C$
18. $\int sinhx\, dx = \cos hx + C$
19. $\int coshx\, dx = \sin h\, x + C$

Example:

Evaluate $\int \frac{x^3 + 2x+1}{x^4} dx$

Solution:

Given $\int \frac{x^3 + 2x+1}{x^4} dx$

$= \int \left(\frac{1}{x} + \frac{2}{x^3} + \frac{1}{x^4}\right) dx \quad = \int \left(\frac{1}{x} + 2x^{-3} + x^{-4}\right) dx$

$= \log x + 2\frac{x^{-2}}{(-2)} + \frac{x^{-3}}{(-3)} + C$

$= \log x - \frac{1}{x^2} - \frac{1}{3x^3} + C$

Example:

Evaluate $\int \frac{x^3 - 2\sqrt{x}}{x} dx$

Solution:

Given $\int \frac{x^3 - 2\sqrt{x}}{x} dx$

$= \int \left(x^2 - \frac{2}{\sqrt{x}}\right) dx \quad = \int (x^2 - 2x^{-1/2}) dx$

$= \frac{x^3}{3} - 2\frac{x^{1/2}}{1/2} + C = \frac{1}{3}x^3 - 4\sqrt{x} + C$

Example:

Evaluate $\int (x^{2/5} - x^{-3/5})^2 \, dx$

Solution:

Given $\int (x^{2/5} - x^{-3/5})^2 dx$

$= \int [(x^{2/5})^2 + (x^{-3/5})^2 - 2(x^{2/5})(x^{-3/5}) \, dx]$

$= \int [x^{4/5} + x^{-6/5} - 2(x^{-1/5}) \, dx]$

$= \frac{x^{\frac{4}{5}+1}}{(\frac{4}{5}+1)} + \frac{x^{\frac{-6}{5}+1}}{(-\frac{6}{5}+1)} - \frac{x^{\frac{-1}{5}+1}}{(-\frac{1}{5}+1)} + C$

$= \frac{5}{9}x^{9/5} - 5x^{-1/5} - \frac{5}{2}x^{4/5} + C$

Example:

Evaluate $\int x^2 (1-x)^2 \, dx$

Solution:

Given $\int x^2 (1-x)^2 dx$

$= \int x^2 (1 + x^2 - 2x) \, dx$

$= \int (x^2 + x^4 - 2x^3) \, dx$

$$= \frac{x^3}{3} + \frac{x^5}{5} - 2\frac{x^4}{4} + C$$

Example:

Evaluate $\int \frac{1}{1+\sin x}\, dx$

Solution:

Given $\int \frac{1}{1+\sin x}\, dx$

$$\int \frac{1}{1+\sin x}\, dx = \int \frac{1}{1+\sin x}\,\frac{1-\sin x}{1-\sin x}\, dx$$

$$= \int \frac{1-\sin x}{1-\sin^2 x}\, dx = \int \frac{1-\sin x}{\cos^2 x}\, dx$$

$$= \int [\sec^2 x - \sec x \tan x]\, dx \qquad [\because \frac{1}{\cos x} = \sec x\,;\ \frac{\sin x}{\cos x} = \tan x]$$

$$= \tan x - \sec x + C$$

Example:

Evaluate $\int \frac{\sin^2 x}{1+\cos x}\, dx$

Solution:

Given $\int \frac{\sin^2 x}{1+\cos x}\, dx = \int \frac{1-\cos^2 x}{1+\cos x}\, dx \qquad [\because \sin^2 x = 1 - \cos^2 x]$

$$= \int \frac{(1-\cos x)\,(1+\cos x)}{(1+\cos x)}\, dx \qquad [\because a^2 - b^2 = (a-b)\,(a+b)]$$

$$= \int (1-\cos x)\, dx$$

$$= x - \sin x + C$$

Substitution Rule

Substitution Rule:

Let us see the suitable substitution to convert the given integral into a standard form.

The integrand of the form

(i) $\int F(f(x))\, f'(x)dx$ (ii) $\int (f(x))^n f'(x)dx$

(iii) $\int \frac{f'(x)}{(f(x))^n}dx$ (iv) $\int \frac{f'(x)}{F(f(x))}dx$

(v) $\int \frac{e^{f(x)}}{f'(x)}dx$ (vi) $\int e^{f(x)} f'(x)dx$

Substitute $u = f(x) \therefore du = f'(x)$ and then proceed.

Algebraic functions:

Example:

(i) Evaluate $\int \sqrt{2x+1}\, dx$.

Solution:

Put $u = 2x + 1 \Rightarrow du = 2dx \Rightarrow dx = \frac{du}{2}$

$$\int \sqrt{2x+1}\, dx = \int \sqrt{u}\frac{du}{2} = \frac{1}{2}\left[\frac{u^{3/2}}{3/2}\right] + C$$

$$= \frac{2}{2\times 3}(u)^{3/2} + C = \frac{(2x+1)^{3/2}}{3} + C$$

(ii)Evaluate $\int \frac{1}{(ax+b)^4}dx$.

Solution:

Put $u = ax + b \Rightarrow du = adx \Rightarrow dx = \frac{du}{a}$

$$\int \frac{1}{(ax+b)^4}dx = \int \frac{1}{u^4}\frac{du}{a}$$

$$= \frac{1}{a}\int u^{-4}du$$

$$= \frac{1}{a}\left[\frac{u^{-3}}{-3}\right] + C$$

$$= \frac{-1}{3a}\left[\frac{1}{u^3}\right] + C = \frac{-1}{3a}\left[\frac{1}{(ax+b)^3}\right] + C$$

(iii) Evaluate $\int x^5\sqrt{x^2+1}dx$.

Solution:

Put $u = x^2 + 1 \Rightarrow x^2 = u - 1;$ $du = 2xdx \Rightarrow xdx = \frac{du}{2}$

$$\int x^5\sqrt{x^2+1}dx = \int x^4 \sqrt{x^2+1}\, x\, dx$$

$$= \int \sqrt{u}\,(u-1)^2 \frac{du}{2}$$

$$= \frac{1}{2}\int \sqrt{u}\,(u^2 - 2u + 1)\, du$$

$$= \frac{1}{2} \int (u^{5/2} - 2\, u^{3/2} + u^{1/2})\; du$$

$$= \frac{1}{2} \left(\frac{u^{7/2}}{7/2} - \frac{2u^{5/2}}{5/2} - \frac{u^{3/2}}{3/2}\right) + C$$

$$= \left(\frac{u^{7/2}}{7} - \frac{2u^{5/2}}{5} - \frac{u^{3/2}}{3}\right) + C$$

$$= \left(\frac{(x^2+1)^{7/2}}{7} - \frac{2(x^2+1)^{5/2}}{5} - \frac{(x^2+1)^{3/2}}{3}\right) + C$$

(iv) Evaluate $\int \frac{x^2}{\sqrt{x+5}}\, dx$

Solution:

Given $\int \frac{x^2}{\sqrt{x+5}}\, dx$

$$Put\ u = \sqrt{x+5} \quad \Rightarrow \quad du = \frac{1}{2\sqrt{x+5}} dx$$

$$\Rightarrow \quad 2du = \frac{1}{\sqrt{x+5}} dx$$

$$u^2 = x + 5 \quad \Rightarrow x = u^2 - 5 \quad \Rightarrow x^2 = (u^2 - 5)^2 = u^4 - 10u^2 + 25$$

$$\int \frac{x^2}{\sqrt{x+5}}\, dx \quad = \int (u^4 - 10u^2 + 25)\; 2\; du = 2\int (u^4 - 10u^2 + 25) du$$

$$= 2\left[\frac{u^5}{5} - 10\frac{u^3}{3} + 25u\right] + C$$

$$= \frac{2}{5}(x+5)^{5/2} - \frac{20}{3}(x+5)^{3/2} + 50(x+5)^{1/2} + C$$

(v) Evaluate $\int \frac{1}{\sqrt{x}(1+\sqrt{x})^2}\, dx$

Solution:

Given $\int \frac{1}{\sqrt{x}(1+\sqrt{x})^2}\, dx$

$$Put\ u = 1 + \sqrt{x} \quad \Rightarrow du = \frac{1}{2\sqrt{x}} dx \qquad \Rightarrow 2du = \frac{1}{\sqrt{x}} dx$$

$$= \int \frac{1}{u^2}\, 2\, du \quad = 2\int u^{-2}\, du \quad = 2\left(\frac{u^{-1}}{-1}\right) + C$$

$$= -\frac{2}{u} + C$$

$$= -\frac{2}{1+\sqrt{x}} + C$$

(vi) Evaluate $\int \frac{(1+\sqrt{x})^{1/3}}{\sqrt{x}}\, dx$

Solution:

Given $\int \frac{(1+\sqrt{x})^{1/3}}{\sqrt{x}} dx$

$$Put \quad u = 1+\sqrt{x} \quad \Rightarrow du = \frac{1}{2\sqrt{x}}dx \quad \Rightarrow 2du = \frac{1}{\sqrt{x}}dx$$

$$\int \frac{(1+\sqrt{x})^{1/3}}{\sqrt{x}}dx = \int u^{1/3}\, 2\,du = 2\int u^{1/3}\,du = 2\,\frac{u^{4/3}}{(4/3)} + C$$

$$= \frac{3}{2}(u)^{4/3} + C$$

$$= \frac{3}{2}(1+\sqrt{x})^{4/3} + C$$

Logarithmic functions:

Example :

(i) Evaluate $\int \frac{logx}{x}\, dx$

Solution:

Given $\int \frac{logx}{x}\, dx$

Put $u = logx \Rightarrow du = \frac{1}{x}dx$

$$\int \frac{logx}{x}\, dx = \int u du = \frac{u^2}{2} + C = \frac{(logx)^2}{2} + C$$

(ii)Evaluate: $\int \frac{(logx)^2}{x} dx$

Solution:

Given $\int \frac{(logx)^2}{x} dx$

Put $u = logx \quad \Rightarrow du = \frac{1}{x}dx$

$$\int \frac{(logx)^2}{x}\, dx = \int u^2 du = \frac{u^3}{3} + C = \frac{(logx)^3}{3} + C$$

(iii)Evaluate $\int \frac{\sin(2+logx)}{x}\, dx$

Solution:

Given $\int \frac{\sin(2+logx)}{x} dx$

Put $u = 2 + logx \quad \Rightarrow du = \frac{1}{x}dx$

$$\int \frac{\sin(2+logx)}{x}dx = \int \sin u\, du = -\cos u + C$$

$$= -\cos(2+logx) + C$$

(iv) Evaluate $\int \frac{dx}{x\sqrt{logx}}$

Solution:

Given $\int \frac{dx}{x\sqrt{logx}}$

Put $u = logx \quad \Rightarrow du = \frac{1}{x}dx$

$$\int \frac{dx}{x\sqrt{logx}} = \int \frac{du}{\sqrt{u}} = 2\sqrt{u} + C \quad = 2\sqrt{logx} + C$$

(v)Evaluate $\int secx \log(secx + tanx)\, dx$

Solution:

Given $\int secx \log(secx + tanx)\, dx$

Put $u = \log(secx + tanx) \quad \Rightarrow \; du = \frac{1}{(secx+tanx)}(secxtanx + sec^2x)dx$

$$\Rightarrow du = \frac{sec(tanx+secx)}{(secx+tanx)}dx$$

$$= \int udu = \frac{u^2}{2} + C$$

$$= \frac{1}{2}[log(secx + tanx)]^2 + C$$

Exponential functions

Example:

(i)Evaluate $\int e^{cosx} sinx\, dx$

Solution:

Given $\int e^{cosx} sinx\, dx$

$Put\ u = e^{cosx} \quad \Rightarrow \quad du = e^{cosx}(-sinx)dx$

$$\int e^{cosx} sinx\, dx = \int (-du) \quad = -\int du \quad = -u + C \quad = -e^{cosx} + C$$

(ii)Evaluate $\int e^{x^3} x^2 dx$

Solution:

Given $\int e^{x^3} x^2 dx$

$Put\ u = e^{x^3} \quad \Rightarrow \quad du = e^{x^3} 3x^2 dx \quad \Rightarrow \quad \frac{du}{3} = e^{x^3} x^2 dx$

$$\int e^{x^3} x^2 dx = \int \frac{du}{3} \quad = \frac{1}{3}\int du \quad = \frac{1}{3}u + C \quad = \frac{1}{3}e^{x^3} + C$$

(iii) Evaluate $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$

Solution:

Given $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$

$Put\ u = e^{\sqrt{x}} \quad \Rightarrow \quad du = e^{\sqrt{x}} \frac{1}{2\sqrt{x}} dx \quad \Rightarrow \quad 2du = \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx = \int 2du \quad = 2\int du \quad = 2u + C \quad = 2e^{\sqrt{x}} + C$$

(iv) Evaluate $\int \frac{e^{tan^{-1}x}}{1+x^2}\, dx$

Solution:

Given $\int \frac{e^{tan^{-1}x}}{1+x^2}\, dx$

Put $u = tan^{-1}x$ $\qquad du = \frac{1}{1+x^2}\, dx$

$$\int \frac{e^{tan^{-1}x}}{1+x^2}\, dx = \int e^u du = e^u + C$$

$$= e^{tan^{-1}x} + C$$

(v) Evaluate $\int \frac{1}{e^x+e^{-x}}\, dx$

Solution:

Given $\int \frac{1}{e^x+\frac{1}{e^x}}\, dx = \int \frac{e^x dx}{e^{2x}+1}$

Put $e^x = u \Rightarrow e^x dx = du$

$$= \int \frac{du}{u^2+1}$$

$$= tan^{-1}u + C \quad = tan^{-1}e^x + C$$

Techniques of integration

Integration by parts

If the integrand is either a product or quotient of polynomial and a transcendental function such as trigonometric, exponential or logarithmic function then have to develop different methods to evaluate them.

$$\int f(x)g'(x)dx = f(x)\, g(x) - \int g(x)f'(x)dx$$

The above formula is called the integration by parts.

Let if we take $u = f(x)$ and $v = g(x)$

Then the above formula becomes $\int udv = uv - \int vdu$

To choose u, we should follow the following order

I – Inverse function

L – Logarithmic function

A – Algebraic function

T – Trigonometric function

E – Exponential function

Note:

The generalized integration by parts formula is known as Bernoulli's formula

Bernoulli's formula states that

$$\int udv = uv - u'v_1 + u''v_2 - u'''v_3 + \cdots$$

Where $v_1, v_2, v_3, \ldots$ are functions obtained by integrating v successively with respect to x and $u', u'', \ldots$ are functions obtained by differentiating u successively with respect to x.

Example:

Evaluate $\int xe^{-x}dx$

Solution:

Let $u = x$ $\quad dv = e^{-x}dx$

$du = dx$ $\quad v = -e^{-x}$

$$\int udv = uv - \int vdu$$

$$\int xe^{-x}dx = x(-e^{-x}) - \int -e^{-x}\, dx$$

$$= -xe^{-x} + \int e^{-x}\, dx$$

$$= -xe^{-x} + (-e^{-x}) + C = -(xe^{-x} + e^{-x} + C)$$

$$= -e^{-x}(x + 1) + C$$

Example :

Evaluate $\int x^4 logx dx$

Solution:

Let $u = logx \qquad dv = x^4 dx$

$du = \frac{1}{x} dx \quad v = \int x^4 dx = \frac{x^5}{5}$

$$\int udv = uv - \int vdu$$

$$\int x^4 logx dx = (logx)\left(\frac{x^5}{5}\right) - \int \frac{x^5}{5}\frac{1}{x} dx$$

$$= \frac{x^5}{5} logx - \frac{1}{5}\int x^4 dx$$

$$= \frac{x^5}{5} logx - \frac{1}{5}\frac{x^5}{5} + C = \frac{x^5}{5} logx - \frac{x^5}{25} + C$$

Example :

Evaluate $\int (\log x)^2$ dx

Solution:

Let $u = (\log x)^2 \qquad dv = dx$

$du = 2\, logx \left(\frac{1}{x}\right) dx \qquad v = \int dx = x$

$$\int udv = uv - \int vdu$$

$$\int (\log x)^2 dx = (\log x)^2 x - \int \left(x\, 2\, logx \left(\frac{1}{x}\right) dx\right)$$

$$= x(\log x)^2 - 2\int logx dx \ldots (1)$$

Take $\int logx dx$

Let $u = logx \qquad dv = dx$

$du = \frac{1}{x} dx \qquad v = \int dx = x$

$$\int udv = uv - \int vdu$$

$$\int logx dx = (logx)(x) - \int x \frac{1}{x} dx = xlogx - \int dx = x\,logx - x$$

$$(1) \Rightarrow \int (\log x)^2 dx = x(\log x)^2 - 2\,[xlogx - x] + C$$

Example :

Evaluate $\int x sec^2 2x dx$

Solution:

Let $u = x \qquad dv = sec^2 2x dx$

$du = dx \quad v = \int sec^2 2x dx = \frac{\tan 2x}{2}$

$$\int udv = uv - \int vdu$$

$$\int xsec^2 2xdx = (x)\left(\frac{\tan 2x}{2}\right) - \int \frac{\tan 2x}{2} dx$$

$$= \frac{1}{2} x \tan 2x - \frac{1}{2} \int \tan 2x dx$$

$$= \frac{1}{2} x \tan 2x - \frac{1}{2}\left[\frac{\log(\sec 2x)}{2}\right] + C$$

$$= \frac{1}{2} x \tan 2x - \frac{1}{4} \log(\sec 2x) + C$$

Example :

Evaluate $\int xsin^2 xdx$

Solution:

Let $u = x$ $\quad dv = sin^2 xdx$

$du = dx$ $\quad v = \frac{1}{2}\int (1 - cos2x)dx = \frac{1}{2}\left(x - \frac{sin2x}{2}\right)$

$$\int udv = uv - \int vdu$$

$$\int xsin^2 xdx = \frac{x}{2}\left(x - \frac{sin2x}{2}\right) - \frac{1}{2}\int \left(x - \frac{sin2x}{2}\right) dx$$

$$= \frac{x^2}{2} - \frac{xsin2x}{4} - \frac{1}{2}\int \left(x - \frac{sin2x}{2}\right) dx$$

$$= \frac{x^2}{2} - \frac{xsin2x}{4} - \frac{1}{2}\left(\frac{x^2}{2} + \frac{cos2x}{4}\right) + C$$

$$= \frac{x^2}{2} - \frac{xsin2x}{4} - \frac{x^2}{4} - \frac{cos2x}{8} + C$$

$$= \frac{x^2}{4} - \frac{xsin2x}{4} - \frac{cos2x}{8} + C$$

Example :

Evaluate $\int \frac{x}{1+cosx} dx$

Solution:

$$\int \frac{x}{1+cosx} dx = \int \frac{x}{2cos^2\frac{x}{2}} dx \qquad \left[\because 1 + cosx = 2cos^2\frac{x}{2}\right]$$

$$= \frac{1}{2}\int xsec^2\frac{x}{2} dx \cdots (1)$$

Let $u = x$ $\quad dv = sec^2\frac{x}{2} dx$

$du = dx$ $\quad v = \int sec^2\frac{x}{2} dx = \frac{\tan(\frac{x}{2})}{\frac{1}{2}} = 2\tan\frac{x}{2}$

$$\int udv = uv - \int vdu$$

$$(1) \Rightarrow \int \frac{x}{1+cosx} dx = \frac{1}{2}\left[x\left(2\tan\frac{x}{2}\right) - \int 2\tan\frac{x}{2} dx\right]$$

$$= x\tan\frac{x}{2} - \frac{log[\sec(\frac{x}{2})]}{\frac{1}{2}} + C$$

$$= x\tan\frac{x}{2} - 2\log\left[\sec\left(\frac{x}{2}\right)\right] + C$$

Example :

Evaluate $\int \frac{x}{1+sinx} dx$

Solution:

$$\int \frac{x}{1+sinx}dx = \int \frac{x\,(1-sinx)}{(1+sinx)(1-sinx)}dx$$

$$= \int \frac{x\,(1-sinx)}{1-sin^2x}dx = \int \frac{x\,(1-sinx)}{cos^2x}dx$$

$$= \int (xsec^2x - xsecxtanx)\,dx$$

$$= \int xsec^2xdx - \int xsecxtanxdx \quad \ldots (1)$$

Take $\int xsec^2xdx$

Let $u = x$ $\quad\quad dv = sec^2xdx$

$du = dx$ $\quad\quad v = \int sec^2xdx = tanx$

$$\int udv = uv - \int vdu$$

$$\int xsec^2xdx = (x)(tanx) - \int tanxdx$$

$$= xtanx - \log(secx) \ldots (2)$$

Take $\int xsecxtanxdx$

Let $u = x$ $\quad\quad dv = secxtanxdx$

$du = dx$ $\quad\quad v = \int secxtanxdx = secx$

$$\int udv = uv - \int vdu \quad = \int xsecxtanxdx = (x)(secx) - \int secxdx$$

$$= xsecx - \log(secx + \tan x) \quad \ldots (3)$$

(1) $\Rightarrow \int \frac{x}{1+sinx}dx = xtanx - \log(secx) - xsecx + \log(secx + tanx) + C$ $\quad [\because$ $by(2)and(3)\,]$

Solution:

Let $u = x^2 + 2x\,,\quad u' = 2x + 2,\quad u'' = 2,\quad u''' = 0$

$dv = cosxdx,\quad v = sinx,\quad v_1 = -cosx,\quad v_2 = -sinx$

$$\int udv = uv - u'v_1 + u''v_2 - \cdots$$

$$\int (x^2 + 2x)\,cosxdx = (x^2 + 2x)sinx - (2x + 2)(-cosx) + (2)(-sinx) + C$$

$$= (x^2 + 2x - 2)sinx + (2x + 2)(cosx) + C$$

Example :

Evaluate $\int (x^2 e^{2x})\, dx$

Solution:

Let $u = x^2$, $u' = 2x$, $u'' = 2$,

$dv = e^{2x}dx$, $v = \frac{e^{2x}}{2}$, $v_1 = \frac{e^{2x}}{4}$, $v_2 = \frac{e^{2x}}{8}$

$$\int udv = uv - u'v_1 + u''v_2 - \cdots$$

$$\int (x^2 e^{2x})\, dx = (x^2)\frac{e^{2x}}{2} - (2x)\frac{e^{2x}}{4} + (2)\frac{e^{2x}}{8} + C$$

$$= (x^2)\frac{e^{2x}}{2} - (x)\frac{e^{2x}}{2} + \frac{e^{2x}}{4} + C$$

Example :

Evaluate $\int e^x cosxdx$

Solution:

Let $u = e^x$ $dv = cosxdx$

$du = e^x dx$ $v = \int cosxdx = sinx$

$$\int udv = uv - \int vdu$$

$$I = \int e^x cosxdx = e^x sinx - \int sinxe^x dx \quad \ldots (1)$$

Take $\int e^x sinxdx$

Let $u = e^x$ $dv = sin\, xdx$

$du = e^x dx$ $v = \int sinxdx = -cosx$

$$\int udv = uv - \int vdu$$

$$\int e^x sinxdx = (e^x)(-cosx) - \int (-cosx)(e^x)\, dx$$

$$= -e^x cosx + \int e^x cosxdx = -e^x cosx + I$$

$$(1) \Rightarrow I = e^x sinx - [-e^x cosx + I] + C$$

$$I = e^x sinx + e^x cosx - I + C$$

$$2I = e^x sinx + e^x cosx + C$$

$$I = \frac{1}{2}[e^x sinx + e^x cosx] + C$$

$$\therefore \int e^x cosxdx = \frac{e^x}{2}[sinx + cosx] + C$$

Example :

Evaluate $\int e^{2x} sinx dx$

Solution:

$$I = \int e^{2x} sinxdx \quad \dots(1)$$

Let $u = sinx$ $\qquad dv = e^{2x}dx$

$du = cosx\, dx$ $\qquad v = \frac{e^{2x}}{2}$

$$\int udv = uv - \int vdu$$

$$I = sinx\frac{e^{2x}}{2} - \int \frac{e^{2x}}{2} cosx\, dx = \frac{e^{2x}}{2} sinx - \frac{1}{2} I_1 \dots(2)$$

Take $I_1 = \int e^{2x} \cos x\, dx$

Let $u = cosx$ $\qquad dv = e^{2x}dx$

$du = -sinxdx$ $\qquad v = \frac{e^{2x}}{2}$

$$I_1 = cosx\frac{e^{2x}}{2} - \int \frac{e^{2x}}{2}(-sinx)dx$$

$$= \frac{e^{2x}}{2} cosx + \frac{1}{2} \int e^{2x} sinxdx$$

$$= \frac{e^{2x}}{2} cosx + \frac{1}{2} I$$

$$(2) \Rightarrow I = \frac{e^{2x}}{2} sinx - \frac{1}{2}[\frac{e^{2x}}{2} cosx + \frac{1}{2} I]$$

$$I = \frac{e^{2x}}{2} sinx - \frac{e^{2x}}{4} cosx - \frac{1}{4} I$$

$$I + \frac{1}{4} I = \frac{e^{2x}}{2} sinx - \frac{e^{2x}}{4} cosx$$

$$\frac{5}{4} I = \frac{e^{2x}}{4}(2\, sinx - cosx)$$

$$\therefore I = \frac{e^{2x}}{5}(2\, sinx - cosx) + C$$

Example :

Evaluate $\int tan^{-1}xdx$. Also find $\int_0^1 tan^{-1}xdx$

Solution:

Let $u = tan^{-1}x$ $\qquad dv = dx$

$du = \frac{1}{1+x^2}dx$ $\quad v = \int dx = x$

$$\int udv = uv - \int vdu$$

$$\int tan^{-1}xdx = xtan^{-1}x - \int x\left(\frac{1}{1+x^2}\right) dx$$

$$= xtan^{-1}x - \int \left(\frac{x}{1+x^2}\right) dx \dots(1)$$

Take $\int \left(\frac{x}{1+x^2}\right) dx$

Put $t = 1 + x^2$, $dt = 2xdx$

$$\int \left(\frac{x}{1+x^2}\right) dx = \int \frac{1}{t}\frac{1}{2}dt = \frac{1}{2}\int \frac{1}{t}dt = \frac{1}{2}logt = \frac{1}{2}\log(1 + x^2)$$

$$(1) \Rightarrow \int tan^{-1}xdx = xtan^{-1}x - \frac{1}{2}\log(1 + x^2) + C \text{ ... } (2)$$

To find $\int_0^1 tan^{-1}x$

$$(2) \Rightarrow \int_0^1 tan^{-1}x = \left[xtan^{-1}x\right]_0^1 - \left[\frac{1}{2}\log(1 + x^2)\right]_0^1$$

$$= tan^{-1}1 - 0 - [\frac{1}{2}log2 - \frac{1}{2}log1]$$

$$= \frac{\pi}{4} - \frac{1}{2}log2 \qquad [\because \log 1 = 0]$$

Reduction Formula

(I) Find the reduction formula for $\int sin^n x\, dx$; $n \geq 2$ is an integer

Solution:

Consider $I_n = \int sin^n x\, dx = \int sin^{n-1}x\, sinx\, dx$

We know by the method of integration by part

$$\int u\, dv = uv - \int v\, du$$

Let $u = sin^{n-1}x$ $\qquad dv = sinx\, dx;$

$du = (n-1)sin^{n-2}x\, cosx\, dx \qquad v = \int sinx\, dx = -cosx$

$$I_n = -cosx\, sin^{n-1}x - \int(-cosx)(n-1)sin^{n-2}x\, cosx\, dx$$

$$= -cosx\, sin^{n-1}x + (n-1) \int cos^2 x sin^{n-2}x\ dx$$

$$= -cosx\, sin^{n-1}x + (n-1) \int(1 - sin^2x)\, sin^{n-2}x\ dx$$

$$= -cosx\, sin^{n-1}x + (n-1)\int sin^{n-2}x\ dx - (n-1)\int sin^n x\, dx$$

$$= -cosx\, sin^{n-1}x + (n-1)\int sin^{n-2}x\ dx - (n-1)I_n$$

$$I_n + (n-1)I_n = -cosx\, sin^{n-1}x + (n-1)\int sin^{n-2}x\ dx$$

$$nI_n = -cosx\, sin^{n-1}x + (n-1)\int sin^{n-2}x\ dx$$

$$I_n = -\frac{cosx\, sin^{n-1}x}{n} + \frac{(n-1)}{n}\int sin^{n-2}x\ dx$$

The ultimate integral is I_0 or I_1

n even: $I_0 = \int dx = x + C$ [Put n = 0 in (1)]

n odd: $I_1 = \int sinx\, dx = -cosx + C$ [Put n = 1 in (1)]

(II) **Find the reduction formula for** $\int cos^n x\, dx$; $n \geq 2$ is an integer

Solution:

Consider $I_n = \int cos^n x\, dx = \int cos^{n-1}x\, cosx\, dx$

We know by the method of integration by part

$$\int u\, dv = uv - \int v\, du$$

Let $u = cos^{n-1}x$ $\qquad dv = cosx\, dx$

$du = (n-1)cos^{n-2}x\,(-sinx)dx \qquad v = \int cosx\, dx = sinx$

$$I_n = sinx\, cos^{n-1}x - \int (sinx)[-(n-1)cos^{n-2}x\, sinx]\, dx$$

$$= sinx\, cos^{n-1}x + (n-1)\int sin^2 x cos^{n-2}x\; dx$$

$$= sinx\, cos^{n-1}x + (n-1)\int (1 - cos^2x)\, cos^{n-2}x\; dx$$

$$= sinx\, cos^{n-1}x + (n-1)\int cos^{n-2}x\; dx - (n-1)\int cos^n x\, dx$$

$$= sinx\, cos^{n-1}x + (n-1)\int cos^{n-2}x\; dx - (n-1)I_n$$

$$I_n + (n-1)I_n = sinx\, cos^{n-1}x + (n-1)\int cos^{n-2}x\; dx$$

$$nI_n = sinx\, cos^{n-1}x + (n-1)\int cos^{n-2}x\; dx$$

$$I_n = \frac{sinx\, cos^{n-1}x}{n} + \frac{(n-1)}{n}\int cos^{n-2}x\; dx$$

The ultimate integral is I_0 or I_1

n even: $I_0 = \int dx = x + C$ [Put n = 0 in (1)]

n odd: $I_1 = \int cosx\, dx = sinx + C$ [Put n = 1 in (1)]

(III) Find the reduction formula for $\int_0^{\pi/2} sin^n x\, dx$

Solution:

Consider $I_n = \int_0^{\pi/2} sin^n x\, dx$

We know that $\int sin^n x\, dx = -\frac{cosx\, sin^{n-1}x}{n} + \frac{n-1}{n}\int sin^{n-2}x\; dx$

$$\int_0^{\pi/2} sin^n x\, dx = \left[-\frac{cosx\, sin^{n-1}x}{n}\right]_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2} sin^{n-2} x\, dx$$

$$= 0 + \frac{n-1}{n}\int_0^{\pi/2} sin^{n-2} x\, dx$$

$$= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\int_0^{\pi/2} sin^{n-4} x\, dx$$

$$= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\int_0^{\pi/2} sin^{n-6} x\, dx$$

$$= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdot\frac{n-7}{n-6}\cdots I$$

If n is even then,

$$I = \int_0^{\pi/2} dx = (x)_0^{\pi/2} = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

If n is odd then,

$$I = \int_0^{\pi/2} sinx\, dx = (-cosx)_0^{\pi/2} = -cos\frac{\pi}{2} + cos0 = 0 + 1 = 1$$

Thus,

$$\int_0^{\pi/2} sin^n x\, dx = \begin{cases} \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdot\frac{n-7}{n-6}\cdots\frac{1}{2}\cdot\frac{\pi}{2}, & if\ n\, is\ even \\ \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdot\frac{n-7}{n-6}\cdots\frac{2}{3}\cdot 1, & if\ n\, is\ odd \end{cases}$$

(IV) Find the reduction formula for $\int_0^{\pi/2} cos^n x\, dx$

Solution:

Consider $I_n = \int_0^{\pi/2} cos^n x\, dx$

We know that $\int cos^n x\, dx = \frac{sinx\, cos^{n-1}x}{n} + \frac{(n-1)}{n}\int cos^{n-2}x\ dx$

$$\int_0^{\pi/2} cos^n x\, dx = \left[\frac{sinx\, cos^{n-1}x}{n}\right]_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2} cos^{n-2} x\, dx$$

$$= 0 + \frac{n-1}{n}\int_0^{\pi/2} cos^{n-2} x\, d$$

$$= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\int_0^{\pi/2} cos^{n-4} x\, dx$$

$$= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\int_0^{\pi/2} cos^{n-6} x\, dx$$

$$= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdot\frac{n-7}{n-6}\cdots I$$

If n is even then,

$$I = \int_0^{\pi/2} dx = (x)_0^{\pi/2} = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

If n is odd then,

$$I = \int_0^{\pi/2} cosx\, dx = (sinx)_0^{\pi/2} = sin\frac{\pi}{2} - sin0 = 1 - 0 = 1$$

Thus,

$$\int_0^{\pi/2} cos^n x\, dx = \begin{cases} \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdot\frac{n-7}{n-6}\cdots\frac{1}{2}\cdot\frac{\pi}{2}, & if\ n\, is\ even \\ \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdot\frac{n-7}{n-6}\cdots\frac{2}{3}\cdot 1, & if\ n\, is\ odd \end{cases}$$

(V) Find the reduction formula for $\int sec^n xdx$, $n \geq 2$ is an integer.

Solution:

Consider $I_n = \int sec^n xdx = \int sec^{n-2} x sec^2 xdx$ ... (1)

We know by the method of integration by part

$$\int u\, dv = uv - \int v\, du$$

Let $u = sec^{n-2}x$ $\qquad dv = sec^2x\,dx$

$du = (n-2)cos^{n-3}x\,(secxtanx)dx \qquad v = \int sec^2x\,dx = tanx$

$I_n = sec^{n-2}x\,tanx - \int(tanx)[(n-2)sec^{n-3}\,x\,secx\,tanx]\,dx$

$= sec^{n-2}x\,tanx - (n-2)\int tan^2\,x\,sec^{n-2}x\,dx$

$= sec^{n-2}x\,tanx - (n-2)\int(sec^2x-1)\,sec^{n-2}x\,dx$

$= sec^{n-2}x\,tanx - (n-2)\int sec^nx\,dx + (n-2)\int sec^{n-2}x\,dx$

$= sec^{n-2}x\,tanx - (n-2)I_n + (n-2)I_{n-2}$

$I_n + (n-2)I_n = sec^{n-2}x\,tanx + (n-2)I_{n-2}$

$(n-1)I_n = sec^{n-2}x\,tanx + (n-2)I_{n-2}$

$$I_n = \frac{1}{n-1}sec^{n-2}x\,tanx + \frac{n-2}{n-1}I_{n-2}$$

The ultimate integral is I_0 or I_1

n even : $I_0 = \int dx = x + C$ [Put n = 0 in (1)]

n odd : $I_1 = \int secx\,dx = \log(secx + tanx) + C$ [Put n = 1 in (1)]

(VI) Find the reduction formula for $\int cosec^n\,xdx$, $n \geq 2$ is an integer.

Solution:

Consider $I_n = \int cosec^n\,xdx = \int cosec^{n-2}\,x\,cosec^2xdx$ ... (1)

We know by the method of integration by part

$$\int u\,dv = uv - \int v\,du$$

Let $u = cosec^{n-2}x$ $\qquad dv = cosec^2x\,dx$

$du = (n-2)sec^{n-3}x\,(-cosecx\,cotx)dx \qquad v = \int cosec^2x\,dx = -cotx$

$I_n = cosec^{n-2}x\,(-cotx) - \int(-cotx)[(n-2)cosec^{n-3}\,x\,(-cosecx\,cotx)]\,dx$

$= -cosec^{n-2}x\,cotx - (n-2)\int cot^2\,x\,cosec^{n-2}x\,dx$

$= -cosec^{n-2}x\,cotx - (n-2)\int(cosec^2x-1)\,cosec^{n-2}x\,dx$

$= -cosec^{n-2}x\,cotx - (n-2)\int cosec^nx\,dx + (n-2)\int cosec^{n-2}x\,dx$

$= -cosec^{n-2}x\,cotx - (n-2)I_n + (n-2)I_{n-2}$

$I_n + (n-2)I_n = -cosec^{n-2}x\,cotx + (n-2)I_{n-2}$

$(n-1)I_n = -cosec^{n-2}x\,cotx + (n-2)I_{n-2}$

$$I_n = -\frac{1}{n-1}cosec^{n-2}x\,cotx + \frac{n-2}{n-1}I_{n-2}$$

The ultimate integral is I_0 or I_1

n even : $I_0 = \int dx = x + C$ [Put n = 0 in (1)]

n odd : $I_1 = \int cosecx\,dx = \log(cosecx - cotx) + C$ [Put n = 1 in (1)]

(VII) Find the reduction formula for $\int cot^n xdx, n \neq 1$

Solution:

Consider $I_n = \int cot^n xdx = \int cot^{n-2}x\, cot^2x\, dx$ ... (1)

$= \int cot^{n-2}x\, (cosec^2x - 1)\, dx$

$= -\int cot^{n-2}x\, (cosec^2x)dx - \int cot^{n-2}x\, dx$

$= -\int cot^{n-2}x\ \ d(cotx) - I_{n-2}$

$= -\frac{1}{n-1}cot^{n-1}x - I_{n-2}$

The ultimate integral is I_0 or I_1

$n\ even$: $I_0 = \int dx = x + C$ [Put n = 0 in (1)]

$n\ odd$: $I_1 = \int cotx\, dx = \log(sinx) + C$ [Put n = 1 in (1)]

(VIII)Find the reduction formula for $\int tan^n xdx, n \neq 1$

Solution:

Consider $I_n = \int tan^n xdx$ ... (1)

$= \int tan^{n-2}x\, tan^2x\, dx$

$= \int tan^{n-2}x\, (sec^2x - 1)\, dx$

$= \int tan^{n-2}x\, sec^2xdx - \int tan^{n-2}x\, dx$

$= \int tan^{n-2}x\ d(tanx) - I_{n-2}$

$= \frac{1}{n-1}tan^{n-1}x - I_{n-2}$

The ultimate integral is I_0 or I_1

$n\ even$: $I_0 = \int dx = x + C$ [Put n = 0 in (1)]

$n\ odd$: $I_1 = \int tanx\, dx = \log(secx) + C$ [Put n = 1 in (1)]

Example:

i) **Evaluate $\int sin^7x\, dx$**

Solution:

Given $\int sin^7x\, dx$

We know that $I_n = -\frac{cosx\, sin^{n-1}x}{n} + \frac{n-1}{n}\int sin^{n-2}x\ \ dx \cdots (1)$

Put n = 7 in equation (1)

$$\int sin^7x\, dx = -\frac{cosx\, sin^{7-1}x}{7} + \frac{7-1}{7}\int sin^{7-2}x\ \ dx$$

$$\int sin^7x\, dx = -\frac{cosx\, sin^6x}{7} + \frac{6}{7}\int sin^5x\ \ dx \cdots (2)$$

Put n = 5 in equation (1)

$$\int sin^5x\,dx = -\frac{cosx\,sin^4x}{5} + \frac{4}{5}\int sin^3x\ \ dx \cdots (3)$$

Put n = 3 in equation (1)

$$\int sin^5x\,dx = -\frac{cosx\,sin^2x}{3} + \frac{2}{3}\int sinx\ \ dx$$

$$= -\frac{cosx\,sin^2x}{3} + \frac{2}{3}(-cosx)$$

$$\therefore (3)\ \ gives \int sin^5x\,dx = -\frac{cosx\,sin^4x}{5} + \frac{4}{5}\left[\frac{-sin^2x\,cosx}{3} - \frac{2}{3}cosx\right]$$

$$= -\frac{cosx\,sin^4x}{5} - \frac{4}{15}sin^2x\,cosx - \frac{8}{15}\ cosx$$

and (2) gives

$$\int sin^7x\,dx = -\frac{cosx\,sin^6x}{7} + \frac{6}{7}\left[\frac{-sin^4x\,cosx}{5} - \frac{4}{15}sin^2\,x\,cosx - \frac{8}{15}\ cosx\right]$$

$$= -\frac{1}{7}cosx\,sin^6x - \frac{6}{35}sin^4x\,cosx - \frac{8}{35} - sin^2x\,cosx - \frac{16}{35}cosx$$

(ii) Evaluate $\int cos^4x\,dx$

Solution:

Given $\int cos^4x\,dx$

We know that $I_n = \frac{sin\,x\,cos^{n-1}x}{n} + \frac{(n-1)}{n}\int\ cos^{n-2}x\ \ dx \cdots (1)$

Put n = 4 in equation (1)

$$\int cos^4x\,dx = \frac{sin\,x\,cos^3x}{4} + \frac{3}{4}\int\ \ cos^2x\ \ dx$$

$$= \frac{sin\,x\,cos^3x}{4} + \frac{3}{4}\int\left(\frac{1+cos\,2x}{2}\right)dx$$

$$= \frac{sin\,x\,cos^3x}{4} + \frac{3}{8}\left(x + \frac{\sin 2x}{2}\right)$$

(iii)Evaluate$\int_0^{\pi/2} sin^7\,x\,dx$

Solution:

Given $\int_0^{\pi/2} sin^7\,x\,dx$

We know that $\int_0^{\pi/2} sin^n\,x\,dx = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdot\frac{n-7}{n-6}\cdots\frac{2}{3}.1\,,\ when\ n\ is\ odd\ \cdots (1)$

Put n = 7 in equation (1)

$$\int_0^{\pi/2} sin^7\,x\,dx = \left(\frac{7-1}{7}\right)\left(\frac{7-3}{7-2}\right)\left(\frac{7-5}{7-4}\right)\ (1)$$

$$= \left(\frac{6}{7}\right)\left(\frac{4}{5}\right)\left(\frac{2}{3}\right)\ (1)$$

(iv)Evaluate $\int_0^{\pi/2} cos^{10}\,x\,dx$

Solution:

Given $\int_0^{\pi/2} cos^{10}\ x\ dx$

We know that $\int_0^{\pi/2} cos^n\ x\ dx = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdot \frac{n-7}{n-6} \cdots \frac{1}{2} \cdot \frac{\pi}{2},\ when\ n\ is\ even\ \cdots (1)$

Put n = 10 in equation (1)

$$\int_0^{\pi/2} cos^n\ x\ dx = \left(\frac{10-1}{10}\right)\left(\frac{10-3}{10-2}\right)\left(\frac{10-5}{10-4}\right)\left(\frac{10-7}{10-6}\right)\left(\frac{10-9}{10}\right)\left(\frac{\pi}{2}\right)$$

$$= \left(\frac{9}{10}\right)\left(\frac{7}{8}\right)\left(\frac{5}{6}\right)\left(\frac{3}{4}\right)\left(\frac{1}{2}\right)\left(\frac{\pi}{2}\right) = \frac{63}{512}\pi$$

(v) Evaluate $\int_0^{\pi} sin^2 x\ dx$

Solution:

Given $\int_0^{\pi} sin^2 x\ dx$

$$\int_0^{\pi} sin^2 x\ dx = \int_0^{\pi} \left(\frac{1-\cos 2x}{2}\right) dx$$

$$= \frac{1}{2}\int_0^{\pi} (1 - cos\ 2x)\ dx$$

$$= \frac{1}{2}\left(x - \frac{\sin 2x}{2}\right)_0^{\pi}$$

$$= \frac{1}{2}\left[\left(\pi - \frac{\sin 2\pi}{2}\right) - \left(0 - \frac{\sin 0}{0}\right)\right]$$

$$= \frac{1}{2}(\pi - 0 - 0 + 0) = \frac{\pi}{2}$$

(vi) Evaluate $\int_0^{\pi/2} sin^{2n+1}\ x\ dx$

Solution:

Given $\int_0^{\pi/2} sin^{2n+1}\ x\ dx$

We know that $\int_0^{\pi/2} sin^n\ x\ dx = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdot \frac{n-7}{n-6} \cdots \frac{2}{3} \cdot 1,\ when\ n\ is\ odd\ \cdots (1)$

Put n = 2 n + 1 in equation (1)

$$\int_0^{\pi/2} sin^n\ x\ dx = \frac{(2n+1)-1}{2n+1} \cdot \frac{(2n+1)-3}{(2n+1)-2} \cdot \frac{(2n+1)-5}{(2n+1)-4} \cdots 1$$

$$= \frac{2n}{2n+1} \cdot \frac{2n-2}{2n-1} \cdot \frac{2n-4}{2n-3} \cdots \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} \cdot 1$$

(vii) Evaluate $\int tan^2\ x\ dx$

Solution:

Given $\int tan^2\ x\ dx$

$$\int tan^2\ x\ dx = \int (sec^2 x - 1)\ dx$$

$$= \int sec^2 x\ dx - \int dx$$

$$= tanx - x + C$$

(viii) Evaluate $\int tan^3\ x\ dx$

Solution:

Given $\int tan^3 x\, dx$

$$\int tan^3 x\, dx = \int tan^2 x\, tanx\, dx$$

$$= \int (sec^2x - 1)\, tanx\, dx$$

$$= \int sec^2x\, tanx\, dx - \int tanx\, dx$$

$$= \int tanx\, d(tanx) - \int tanx\, dx$$

$$= \frac{tan^2x}{2} - \log secx + C$$

(ix) Evaluate $\int_{\pi/6}^{\pi/2} cot^2 x\, dx$

Solution:

Given $\int_{\pi/6}^{\pi/2} cot^2 x\, dx$

$$\int_{\pi/6}^{\pi/2} cot^2 x\, dx = \int_{\pi/6}^{\pi/2} (cosec^2 x - 1)\, dx$$

$$= \int_{\pi/6}^{\pi/2} cosec^2 x\, dx - \int_{\pi/6}^{\pi/2} dx$$

$$= [-\cot x]_{\pi/6}^{\pi/2} - [x]_{\pi/6}^{\pi/2}$$

$$= (-0) - (-\sqrt{3}) - \left(\frac{\pi}{2} - \frac{\pi}{6}\right) = \sqrt{3} - \frac{1}{3}\pi$$

TRIGONOMETRIC INTEGRALS

(I) Products of powers of sines and cosines

Evaluating $\int sin^m x cos^n x dx$

Case (i) If n is odd $(n = 2k + 1)$, then

$$\int sin^m x cos^{2k+1} x dx = \int sin^m x (cos^2 x)^k cosx dx$$
$$= \int sin^m x (1 - sin^2 x)^k cosx dx$$

Here, substitute $u = sinx$

Case (ii) If m is odd $(m = 2k + 1)$, then

$$\int sin^{2k+1} x cos^n x dx = \int (sin^2 x)^k cos^n x sinx dx$$
$$= \int (1 - cos^2 x)^k cos^n x sinx dx$$

Here, substitute $u = cosx$

Note: If both m and n are odd apply case (i) or case (ii)

Case(iii) If both m and n are even, use half- angle identities

$$sin^2 x = \frac{1}{2}(1 - cos2x), \qquad cos^2 x = \frac{1}{2}(1 + cos2x)$$

$$\int_0^{\pi/2} sin^m x cos^n x dx = \frac{m-1}{m+n}\frac{m-3}{m+n-2}\cdots\frac{2}{3+n}\frac{1}{1+n}$$

(if m is odd, n may be even or odd)

$$= \frac{m-1}{m+n}\frac{m-3}{m+n-2}\cdots\frac{1}{2+n}\frac{n-1}{n}\frac{n-3}{n-2}\cdots\frac{2}{3}.1$$

(if m is even, n is odd)

$$= \frac{m-1}{m+n}\frac{m-3}{m+n-2}\cdots\frac{1}{2+n}\frac{n-1}{n}\frac{n-3}{n-2}\cdots\frac{1}{2}\frac{\pi}{2}$$

(if m is even, n is even)

(II) Products of powers of $secx$ and $tanx$

Evaluating $\int tan^m x sec^n x dx$

Case (i) If m is odd $(m = 2k + 1)$, then

$$\int tan^{2k+1} x sec^n x dx = \int (tan^2 x)^k sec^{n-1} x secx tanx dx$$
$$= \int (sec^2 x - 1)^k sec^{n-1} x secx tanx dx$$

Here, substitute $u = secx$

Case (ii) If n is even $(n = 2k)$, then

$$\int tan^m x sec^{2k} x dx = \int tan^m x (sec^2 x)^{k-1} sec^2 x dx$$
$$= \int tan^m x (1 + tan^2 x)^{k-1} sec^2 x dx$$

Here, substitute $u = tanx$

(III) Products of sines and cosines of multiples of x

Evaluating $\int sinmx\, sinnxdx,\quad \int sinmx\, cosnxdx\ and \int cosmx\, cosnx\, dx$

Use the following identities

$$sinmxsinnx = \frac{1}{2}[\cos(m-n)\,x - \cos(m+n)\,x]$$

$$sinmxcosnx = \frac{1}{2}[\sin(m-n)\,x + \sin(m+n)\,x]$$

$$cosmxcosnx = \frac{1}{2}[\cos(m-n)\,x + \cos(m+n)\,x]$$

Example:

(i)Evaluate $\int \boldsymbol{sin^6\, xcos^3xdx}$

Solution:

Given $\int sin^6\, xcos^3xdx$ Here m = 6, n = 3 (odd)

$$= \int sin^6\, xcos^2xcosxdx$$

$$= \int sin^6x(1 - sin^2x)cosxdx \ldots .(1)$$

$$Put\quad u = sinx;\qquad du = cosxdx$$

$$(1) \Rightarrow \int u^6(1-u^2)\,du \quad = \int(u^6 - u^8)du$$

$$= \frac{u^7}{7} - \frac{u^9}{9} + C$$

$$= \frac{sin^7x}{7} - \frac{sin^9x}{9} + C$$

(ii) Evaluate $\int \boldsymbol{sin^2\,(\pi x)cos^5(\pi x)dx}$

Solution:

Given $\int sin^2\,(\pi x)cos^5(\pi x)dx$ (Here m = 2, n = 5 (odd))

$$= \int sin^2\,(\pi x)cos^4(\pi x)\cos(\pi x)\,dx$$

$$= \int sin^2\,(\pi x)[1 - sin^2(\pi x)]^2\cos(\pi x)\,dx \ldots (1)$$

$$Put\ \ u = sin\pi x;\qquad du = \pi cos\pi xdx$$

$$(1) \Rightarrow \int u^2(1-u^2)^2\frac{du}{\pi} \quad = \frac{1}{\pi}\int u^2(1 - 2u^2 + u^4)du$$

$$= \frac{1}{\pi}\int(u^2 - 2u^4 + u^6)du$$

$$= \frac{1}{\pi}[\frac{u^3}{3} - \frac{2u^5}{5} + \frac{u^7}{7}] + C$$

$$= \frac{1}{3\pi}sin^3(\pi x) - \frac{2}{5\pi}sin^5(\pi x) + \frac{1}{7\pi}sin^7(\pi x) + C$$

Example:

Evaluate $\int \boldsymbol{sin^5\, xcos^2xdx}$

Solution:

Given $\int sin^5\, xcos^2xdx$ (Here m = 5 (odd), n = 2)

$$= \int sin^4 x cos^2 x sin x dx$$

$$= \int (1 - cos^2 x)^2 cos^2 x sin x dx \; \ldots . (1)$$

$Put \;\; u = cos x; \quad du = -sin x dx$

$$(1) \Rightarrow \int (1-u^2)^2 u^2 (-du) = -\int (1 - 2u^2 + u^4) u^2 du$$

$$= -\int (u^2 - 2u^4 + u^6) du$$

$$= -[\frac{u^3}{3} - \frac{2u^5}{5} + \frac{u^7}{7}] + C$$

$$= -\frac{1}{3} cos^3(x) + \frac{2}{5} cos^5(x) - \frac{1}{7} cos^7(x) + C$$

Example:

Evaluate $\int cos^2 x sin 2x dx$

Solution:

Given $\int cos^2 x sin 2x dx$

$$= \int 2 sin x cos x cos^2 x dx$$

$= \int 2 sin x cos^3 x dx$ (Here, m = 1, n = 3)

$$= 2 \int sin x cos^3 x dx \ldots (1)$$

$Put \;\; u = cos x; \quad du = -sin x dx$

$$(1) \Rightarrow 2 \int u^3 (-du) = -2 \int u^3 du$$

$$= -2 \frac{u^4}{4} + C = -\frac{1}{2} cos^4 x + c$$

Example:

Evaluate $\int sin^2 x cos^4 x dx$

Solution:

Given $\int sin^2 x cos^4 x dx$ (Here, m = 2, n = 4)

$$= \int \left(\frac{1 - cos 2x}{2}\right) \left(\frac{1 + cos 2x}{2}\right)^2 dx$$

$$= \frac{1}{8} \int (1 - cos 2x)(1 + 2 cos 2x + cos^2 2x) dx$$

$$= \frac{1}{8} \int (1 + cos 2x - cos^2 2x - cos^3 2x) dx$$

$$= \frac{1}{8} [x + \frac{1}{2} sin 2x - \int (cos^2 2x + cos^3 2x) dx] \ldots (1)$$

$$\int cos^2 2x dx = \int \frac{1 + cos 4x}{2} dx = \frac{1}{2}(x + \frac{1}{4} sin 4x)$$

$$\int cos^3 2x dx = \int cos^2 2x cos 2x dx = \int (1 - sin^2 2x) cos 2x dx$$

$Put \;\; u = sin 2x; \quad du = 2 cos 2x dx$

$$\therefore \int cos^3 2x dx = \int (1 - u^2) \frac{du}{2} = \frac{1}{2} [u - \frac{u^3}{3}] = \frac{1}{2} [sin 2x - \frac{1}{3} sin^3 2x]$$

$$(1) \Rightarrow \frac{1}{8}[x + \frac{1}{2}sin2x - \frac{1}{2}x - \frac{1}{8}sin4x - \frac{1}{2}sin2x + \frac{1}{6}sin^3 2x] + C$$

$$= \frac{1}{8}[\frac{1}{2}x - \frac{1}{8}sin4x + \frac{1}{6}sin^3 2x] + C$$

$$= \frac{1}{16}[x - \frac{1}{4}sin4x + \frac{1}{3}sin^3 2x] + C$$

Example:

(i)Evaluate $\int tan\, xsec^3xdx$

Solution:

Given $\int tan\, xsec^3xdx$ (Here m =1 (odd))

$$= \int sec^2x(secx\ tanx)dx$$

$Put\ u = secx;$ $du = secx\ tanxdx$

$$= \int u^2\, du \quad = \frac{u^3}{3} + C = \frac{sec^3x}{3} + C$$

(ii)Evaluate $\int_0^{\pi/3} tan^5xsec^4xdx$

Solution:

Given $\int_0^{\pi/3} tan^5xsec^4xdx$ (Here m = 5 (odd))

$$= \int_0^{\pi/3} tan^4x\ sec^3x\ (secx\ tanx)dx$$

$$= \int_0^{\pi/3} (sec^2x - 1)^2 sec^3x(secx\ tanx)\ dx \quad ...(1)$$

$Put\ u = secx$ when $x = 0 \Rightarrow u = 1$

$du = secx\ tanx\ dx$ $x = \frac{\pi}{3} \Rightarrow u = 2$

$$\therefore (1) \Rightarrow \int_1^2 (u^2 - 1)^2\, u^3\, du = \int_1^2 (u^4 - 2u^2 + 1)u^3\, du$$

$$= \int_1^2 (u^3 - 2u^5 + u^7)\, du$$

$$= [\frac{u^4}{4} - \frac{2u^6}{6} + \frac{u^8}{8}]_1^2$$

$$= (4 - \frac{64}{3} + 32) - (\frac{1}{4} - \frac{1}{3} + \frac{1}{8}) = \frac{117}{8}$$

Example:

(i)Evaluate $\int tan^2xsec^4xd\,x$

Solution:

Given $\int tan^2xsec^4xd\,x$

$$= \int tan^2xsec^2x\ sec^2x\ d\,x$$

$$= \int tan^2x(1 + tan^2x)sec^2x\ dx \ ...(1)$$

$Put\ u = tanx\ ;$ $du = sec^2x\ dx$

$$(1) \Rightarrow \int u^2(1+u^2)\,du = \int (u^2+u^4)du$$
$$= [\frac{u^3}{3}+\frac{u^5}{5}]+C$$
$$= \frac{1}{3}tan^3(x)+\frac{1}{5}tan^5(x)+C$$

(ii)Evaluate $\int tanxsec^2xd\,x$

Solution:

Given $\int tanxsec^2xd\,x$

$$Put\ u = tanx\ ;\quad du = sec^2x\,dx$$
$$= \int udu$$
$$= [\frac{u^2}{2}]+C = \frac{1}{2}tan^2(x)+C$$

Example:

(i) Evaluate $\int sec^3x\,dx$

Solution:

Given I = $\int sec^3x\,dx = \int sec^2x\,secx\,dx$

$$Put\ u = secx \qquad dv = sec^2x\,dx$$
$$du = secx\,tanx\,dx \qquad v = \int sec^2x\,dx = tanx$$
$$\int u\,dv = uv - \int v\,du$$

$$I = (secx)tanx - \int tanx\,(secx\,tanx)dx$$
$$= (secx)tanx - \int tan^2x\,secx\,dx$$
$$= (secx)tanx - \int (sec^2x-1)\,secx\,dx$$
$$= (secx)tanx - \int sec^3x\,dx + \int secx\,dx$$
$$= secx\,tanx - I + \log(secx+tanx)$$
$$2I = secx\,tanx + \log(secx+tanx)$$
$$I = \frac{1}{2}secx\,tanx + \frac{1}{2}\log(secx+tanx)+C$$

(ii)Evaluate $\int tan^2x\,secx\,dx$

Solution:

Given $\int tan^2x\,secx\,dx = \int (sec^2x-1)secx\,dx$

$$= \int sec^3x\,dx - \int secx\,dx$$
$$= \frac{1}{2}secx\,tanx + \frac{1}{2}\log(secx+tanx) - \log(secx+tanx)+C$$

Using example (3.53(i))

$$= \frac{1}{2}secx\,tanx - \frac{1}{2}\log(secx+tanx)+C$$

Example:

(i) Evaluate $\int_0^{\pi/2} \sin^7 x \cos^5 x \, dx$

Solution:

Given $\int_0^{\pi/2} \sin^7 x \ \cos^5 x \, dx$ (Here m = 7 , n = 5)

$$\int_0^{\pi/2} \sin^m x \cos^n x dx = \frac{m-1}{m+n}\frac{m-3}{m+n-2}\cdots\frac{2}{3+n}\frac{1}{1+n}$$ (m is odd, n even or odd)

$$= \frac{7-1}{7+5}\frac{7-3}{7+5-2}\cdots\frac{2}{3+5}\frac{1}{1+5}$$

$$= \left(\frac{6}{12}\right)\left(\frac{4}{10}\right)\left(\frac{2}{8}\right)\left(\frac{1}{6}\right) = \frac{1}{120}$$

(ii)Evaluate $\int_0^{\pi/2} \sin^7 x \, dx$

Solution:

Given $\int_0^{\pi/2} \sin^7 x \, dx$ (Here m = 7 (odd) , n = 0)

$$\int_0^{\pi/2} \sin^m x \cos^n x dx = \frac{m-1}{m+n}\frac{m-3}{m+n-2}\cdots\frac{2}{3+n}\frac{1}{1+n}$$ (m is odd, n even or odd)

$$= \frac{7-1}{7+0}\frac{7-3}{7+0-2}\cdots\frac{2}{3+0}\frac{1}{1+0} = \left(\frac{6}{7}\right)\left(\frac{4}{5}\right)\left(\frac{2}{3}\right)(1) = \frac{16}{35}$$

Example:

i)Evaluate $\int \sin 4x \cos 5x \, dx$

Solution:

Given $\int \sin 4x \cos 5x \, dx$

We know that, $\sin Ax \cos Bx = \frac{1}{2}[\sin(A-B)x + \sin(A+B)x]$

$$= \frac{1}{2}\int[\sin(-x) + \sin 9x]dx$$

$$= \frac{1}{2}\int(-\sin x + \sin 9x)dx$$

$$= \frac{1}{2}\left[\cos x - \frac{1}{9}\cos 9x\right] + C$$

ii)Evaluate $\int \cos 3x \cos 4x \, dx$

Solution:

Given $\int \cos 3x \cos 4x \, dx$

We know that $\cos Ax \cos Bx = \frac{1}{2}[\cos(A-B)x + \cos(A+B)x]$

$$= \frac{1}{2}\int(\cos x + \cos 7x)dx$$

$$= \frac{1}{2}\left[\sin x + \frac{1}{7}\sin 7x\right] + C$$

$$= \frac{1}{2}\sin x + \frac{1}{14}\sin 7x + C$$

iii) Evaluate $\int \sin 5x \sin x \, dx$

Solution:

Given $\int sin\,5x\,sinx\,dx$

We know that $sinAxsinBx = \frac{1}{2}[\cos(A-B)\,x - \cos(A+B)\,x]$

$$= \frac{1}{2}\int(cos4x - cos6x)dx$$

$$= \frac{1}{2}[\frac{1}{4}sin4x - \frac{1}{6}sin6x] + C$$

$$= \frac{1}{2}sin4x - \frac{1}{12}sin6x + C$$

Integration of Rational functions by Partial fraction

Integration of Rational functions by Partial fraction

Let $f(x) = \frac{P(x)}{Q\ (x)}$ be any rational function where $P\ and\ Q$ are polynomials.

If $\deg P < \deg Q$, then f is proper

If $\deg P \geq \deg Q$, then f is improper then to make them proper divide $P(x)\ by\ Q(x)$ by long division until a remainder $R(x)$ is obtained such that $\deg P < \deg Q$

Hence $\frac{P(x)}{Q\ (x)} = S(x) + \frac{R(x)}{Q\ (x)}$ (or) = Quotient $+\frac{Remainder}{Divisor}$

Where $S\ and\ R$ are also polynomials.

Case (i):

The denominator is a product of distinct linear factors

Example:

$$\frac{1}{(x+a)(x+b)} = \frac{A}{(x+a)} + \frac{B}{(x+b)}$$

Case (ii):

The denominator is a product of distinct linear factors, some of which are repeated.

Example:

$$\frac{1}{(x+a)(x+b)^2} = \frac{A}{(x+a)} + \frac{B}{(x+b)} + \frac{C}{(x+b)^2}$$

Case (iii):

The denominator contains irreducible quadratic factors, none of which is repeated.

Example:

$$\frac{1}{(x^2+a)(x^2+b)} = \frac{Ax+B}{(x^2+a)} + \frac{Cx+D}{(x^2+b)}$$

Example:

Evaluate $\int \frac{(x^2+1)}{(x^2-1)(2x+1)}\,dx$

Solution:

$$\frac{(x^2+1)}{(x^2-1)(2x+1)} = \frac{(x^2+1)}{(x-1)(x+1)(2x+1)}$$

$$= \frac{A}{(x-1)} + \frac{B}{(x+1)} + \frac{C}{(2x+1)}$$

$$(x^2 + 1) = A(x + 1)(2x + 1) + B\,(x - 1)(2x + 1) + C\,(x - 1)(x + 1)$$

Put $x = 1$, we get

$2 = A\,(2)\,(3)$

$A = \frac{1}{3}$

Put $x = -1$, we get

$2 = B\,(-2)\,(-1)$

$B = 1$

Put $x = 0$, we get

$1 = A - B - C$

$1 = \frac{1}{3} - 1 - C$

$$C = -2 + \frac{1}{3} = \frac{-5}{3}$$

$$\Rightarrow \frac{(x^2+1)}{(x^2-1)(2x+1)} = \frac{1}{3}\frac{1}{x-1} + \frac{1}{x+1} - \frac{5}{3}\frac{1}{2x+1}$$

$$\int \frac{(x^2+1)}{(x^2-1)(2x+1)}\,dx = \frac{1}{3}\int \frac{1}{x-1}\,dx + \int \frac{1}{x+1}\,dx - \frac{5}{3}\int \frac{1}{2x+1}\,dx$$

$$= \frac{1}{3}\log(x-1) + \log(x+1) - \frac{5}{3}\frac{\log(2x+1)}{2} + C$$

$$= \frac{1}{3}\log(x-1) + \log(x+1) - \frac{5}{6}\log(2x+1) + C$$

Example:

Evaluate $\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx$

Solution:

$$\frac{x^2+2x-1}{2x^3+3x^2-2x} = \frac{x^2+2x-1}{x\,(2x-1)(x+2)} = \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2}$$

$$x^2 + 2x - 1 = A(2x-1)(x+2) + Bx\,(x+2) + Cx\,(2x-1)$$

Put $x = 0$, we get

$$-1 = A\,2$$

$$A = \frac{1}{2}$$

Put $x = \frac{1}{2}$, we get

$$\frac{1}{4} + 1 - 1 = B\left(\frac{1}{2}\right)\left(\frac{5}{2}\right)$$

$$\frac{1}{4} = \frac{5B}{4}$$

$$B = \frac{1}{5}$$

Put $x = -2$, we get

$$4 - 4 - 1 = C(2)(-5)$$

$$-1 = 10\,C$$

$$C = \frac{-1}{10}$$

$$\Rightarrow \frac{x^2+2x-1}{2x^3+3x^2-2x} = \frac{1}{2}\left(\frac{1}{x}\right) + \frac{1}{5}\left(\frac{1}{2x-1}\right) - \frac{1}{10}\left(\frac{1}{x+2}\right)$$

$$\int \frac{x^2+2x-1}{2x^3+3x^2-2x} = \frac{1}{2}\int\left(\frac{1}{x}\right)dx + \frac{1}{5}\int\left(\frac{1}{2x-1}\right)dx - \frac{1}{10}\int\left(\frac{1}{x+2}\right)dx$$

$$= \frac{1}{2}\,logx + \frac{1}{5}\frac{\log(2x-1)}{2} - \frac{1}{10}\log(x+2) + C$$

$$= \frac{1}{2}\,logx + \frac{1}{10}\log\left(\frac{2x-1}{x+2}\right) + C$$

Example:

Evaluate $\int \frac{x^2}{(x-1)^3(x-2)}\,dx$

Solution:

$$\frac{x^2}{(x-1)^3(x-2)} = \frac{A}{x-2} + \frac{B}{x-1} + \frac{C}{(x-1)^2} + \frac{D}{(x-1)^3}$$

$$x^2 = A(x-1)^3 + B(x-1)^2(x-2) + C\,(x-1)(x-2) + D\,(x-2)$$

Put $x = 2$, We get $4 = A$	Equating the coeffs of x^3 On both sides $0 = A + B$, $B = -4$	Put x = 1, we get $1 = D(-1)$, $D = -1$	Put x=0, we get $0 = -A - 2B + 2C - 2D$, $2C = A + 2B + 2D$ $= 4 - 8 - 2$, $C = -3$

$$\Rightarrow \frac{x^2}{(x-1)^3(x-2)} = \frac{4}{x-2} - \frac{4}{x-1} - \frac{3}{(x-1)^2} - \frac{1}{(x-1)^3}$$

$$I = \int \frac{x^2}{(x-1)^3(x-2)}\,dx$$

$$= 4\int \frac{1}{x-2}\,dx - 4\int \frac{1}{x-1}\,dx - 3\int \frac{1}{(x-1)^2}\,dx - \int \frac{1}{(x-1)^3}\,dx$$

$$= 4\log(x-2) - 4\log(x-1) + 3\left(\frac{1}{x-1}\right) + \frac{1}{2(x-1)^2} + C$$

$$= 4\log\left(\frac{x-2}{x-1}\right) + \frac{3}{x-1} + \frac{1}{2(x-1)^2} + C$$

Example:

Evaluate $\int \frac{1}{x^2(x-1)}\,dx$

Solution:

Let $I = \int \frac{1}{x^2(x-1)}\,dx$

$$\frac{1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{(x-1)} \quad \ldots(1)$$

$$1 = Ax(x-1) + B(x-1) + Cx^2$$

Put $x = 0$, We get $1 = -B$, $B = -1$	Put $x = 1$, we get $1 = C$	Equating the Coefficients of x^2 on both side $0 = A + C \Rightarrow A = -C$, $A = -1$

$$(1) \quad \Rightarrow \quad \frac{1}{x^2(x-1)} = \frac{-1}{x} - \frac{1}{x^2} + \frac{1}{(x-1)}$$

$$I = \int \frac{1}{x^2(x-1)}\,dx = -\int \frac{1}{x}\,dx - \int \frac{1}{x^2}\,dx + \int \frac{1}{(x-1)}\,dx$$

$$= -\log x + \frac{1}{x} + \log(x-1) + C = \log\left(\frac{x-1}{x}\right) + \frac{1}{x} + C$$

Example:

Evaluate $\int \frac{10}{(x-1)(x^2+9)}\,dx$

Solution:

Let $I = \frac{10}{(x-1)(x^2+9)}\,dx$

$$\frac{10}{(x-1)(x^2+9)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+9} \qquad \ldots(1)$$

$$10 = A(x^2+9) + (Bx+C)(x-1)$$

Put $x = 1$, We get

$10 = 10A$

A = 1

Equating the Coefficients of x^2 We get

$0 = A + B \Rightarrow B = -A$

$B = -1$

Equating the Coefficients of x,

$0 = -B + C \Rightarrow -B = -C$

$C = -1$

$$(1) \Rightarrow \frac{10}{(x-1)(x^2+9)} = \frac{1}{x-1} + \frac{-x-1}{x^2+9} = \frac{1}{x-1} - \left(\frac{x+1}{x^2+9}\right)$$

$$= \int \frac{1}{x-1}\,dx - \int \frac{x}{x^2+9}\,dx - \int \frac{1}{x^2+9}\,dx$$

$$= \log(x-1) - \frac{1}{2}\log(x^2+9) - \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C$$

Example:

Evaluate $\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\,dx$

Solution:

Let $I = \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}$

$$\begin{array}{r|l} & x+1 \\ \hline x^3-x^2-x+1 & x^4-0x^3-2x^2+4x+1 \\ & x^4-x^3-x^2+x \\ \hline & \quad x^3-x^2+3x+1 \\ & \quad x^3-x^2-x+1 \\ \hline & \qquad 4x \end{array}$$

$$\frac{x^4-2x^2+4x+1}{x^3-x^2-x+1} = x + 1 + \frac{4x+1}{x^3-x^2-x+1}$$

$$= x + 1 + \frac{4x+1}{(x-1)^2(x+1)}$$

$$[x^3 - x^2 - x + 1 = (x-1)^2(x+1)]$$

$$\frac{4x}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x+1)}$$

$$\Rightarrow 4x = A(x+1)(x+1) + B(x+1) + C(x+1)^2$$

Put $x = 1$, We get

$4 = 2B$

B = 2

Put $x = -1$, We get

$-4 = 4C$

$C = -1$

Equating the Coefficient of x^2 on, both sides, we get

$0 = A + C \Rightarrow A = -C$

$A = 1$

$$\Rightarrow \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1} = (x+1) + \frac{1}{x-1} + \frac{2}{(x-1)^2} - \frac{1}{(x+1)}$$

$$I = \int (x+1)dx + \int \frac{1}{x-1}dx + \int \frac{2}{(x-1)^2}dx - \int \frac{1}{(x+1)}dx$$

$$= \frac{x^2}{2} + x + \log(x-1) - \frac{2}{x-1} - \log(x+1) + C$$

$$= \frac{x^2}{2} + x - \frac{2}{x-1} + \log\left(\frac{x-1}{x+1}\right) + C$$

Improper Integrals

The Integral I= $\int_a^b f(x)dx$ is said to be proper or definite only when the limits a and b are finite and the integrand $f(x)$ is continuous in the interval $[a, b]$

Types of Improper Integrals

There are two types of improper integrals

1. With infinite limits of integration
2. The integrand is discontinuous.

Type I (Infinite limits of integration)

1. $\int_a^\infty f(x)dx = \lim_{t\to\infty} \int_a^t f(x)dx$
2. $\int_{-\infty}^b f(x)dx = \lim_{t\to-\infty} \int_t^b f(x)dx$
3. $\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^a f(x)dx + \int_a^\infty f(x)dx$, $'a'$ is a real number.

Provided both the limits on right side exist.

Type II (Discontinuous of the integrand)

1. If f is discontinuous at b, then

$$\int_a^b f(x)dx = \lim_{t\to b^-} \int_a^t f(x)dx$$

2. If f is discontinuous at a, then

$$\int_a^b f(x)dx = \lim_{t\to a^+} \int_t^b f(x)dx$$

3. If f is discontinuous at c, in $[a, b]$ then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

$$= \lim_{t\to c^-} \int_a^t f(x)dx + \lim_{t\to c^+} \int_t^b f(x)dx$$

Provided both the integral's on right exists.

Note:

The improper integral is said to be convergent if the limit exists and is divergent if the limit does not exist.

Example:

Determine whether the integral $\int_1^\infty \frac{1}{x} dx$ is convergent or divergent.

Solution:

The given integral is $\int_1^\infty \frac{1}{x} dx$

an improper integral , since upper limit of integration is infinite then,

$$\int_1^\infty \frac{1}{x} dx = \lim_{t\to\infty} \int_1^t \frac{1}{x} dx$$
$$= \lim_{t\to\infty} [logx]_1^t$$
$$= \lim_{t\to\infty} [logt - log1]$$
$$= \lim_{t\to\infty} [logt - 0] = \infty$$

The given integral is divergent and it diverges to ∞.

Example:

Determine whether the integral $\int_0^\infty \frac{1}{1+x^2} dx$ is convergent or divergent.

Solution:

The given integral is $\int_0^\infty \frac{1}{1+x^2} dx$ an improper integral, since upper limit of integration is infinite then,

$$\int_0^\infty \frac{1}{1+x^2} dx = \lim_{t\to\infty} \int_0^t \frac{1}{1+x^2} dx$$
$$= \lim_{t\to\infty} [tan^{-1}x]_0^t$$
$$= \lim_{t\to\infty} [tan^{-1}t - tan^{-1}0]$$
$$= \lim_{t\to\infty} tan^{-1}t$$
$$= tan^{-1}\infty = \frac{\pi}{2}$$

The given integral is convergent.

Example:

For what values of p the integral $\int_1^\infty \frac{1}{x^p} dx$ convergent?

Solution:

If $p \neq 1$, $\lim_{t\to\infty} \int_1^t \frac{1}{x^p} dx = \lim_{t\to\infty} \int_1^t x^{-p} dx$

$$= \lim_{t\to\infty} \left[\frac{x^{-p+1}}{-p+1}\right]_1^t$$
$$= \lim_{t\to\infty} \left[\frac{t^{-p+1}}{-p+1} - \frac{1}{-p+1}\right]$$

$$= \lim_{t\to\infty} \frac{1}{p-1}\left[1 - \frac{1}{t^{p-1}}\right]$$

$$= \begin{cases} \frac{1}{p-1}, & p > 1, converges \\ \infty, & p \le 1, diverges \end{cases}$$

Example:

Evaluate $\int_1^\infty \frac{logx}{x} dx$

Solution:

Take $I = \int \frac{logx}{x} dx$

Put $u = logx \quad dv = \frac{1}{x} dx \quad du = \frac{1}{x} dx \quad v = logx$

$$I = \int \frac{logx}{x} dx = (logx)^2 - \int logx \left(\frac{1}{x}\right) dx$$

$$I = (logx)^2 - I \Rightarrow 2I = (logx)^2 \Rightarrow I = \frac{1}{2}(logx)^2$$

$$\int_1^\infty \frac{logx}{x} dx = \lim_{t\to\infty} \int_1^t \frac{log}{x} dx = \lim_{t\to\infty} \left(\frac{1}{2}(logx)^2\right)_1^t$$

$$= \lim_{t\to\infty} \left[\frac{1}{2}(logt)^2 - \frac{1}{2}(log1)^2\right]$$

$$= \lim_{t\to\infty} \left[\frac{1}{2}(logt)^2\right] = \infty \quad [\log 1 = 0, \log\infty = \infty]$$

The given integral is divergent.

Example:

Evaluate $\int_{-\infty}^{\infty} xe^{-x^2} dx$

Solution:

Consider $\int xe^{-x^2} dx$

Put $u = x^2, \quad du = 2xdx$

$$\int xe^{-x^2} dx = \int e^{-u} \frac{du}{2} = \frac{1}{2}\left[\frac{e^{-u}}{-1}\right]$$

$$= -\frac{1}{2} e^{-u} = -\frac{1}{2} e^{-x^2} \ldots (1)$$

$$\int_{-\infty}^{\infty} xe^{-x^2} dx = \int_{-\infty}^{0} xe^{-x^2} dx + \int_0^\infty xe^{-x^2} dx \ldots (2)$$

Take $\int_{-\infty}^{0} x e^{-x^2} dx = \lim_{t\to-\infty} \int_t^0 xe^{-x^2} dx = \lim_{t\to-\infty} \left[\frac{-1}{2} e^{-x^2}\right]_t^0$ by (1)

$$= \lim_{t\to-\infty} \left[\frac{-1}{2} + \frac{1}{2} e^{-t^2}\right] = \frac{-1}{2}$$

Take $\int_0^\infty xe^{-x^2}dx = \lim_{t\to\infty}\int_0^t xe^{-x^2}dx = \lim_{t\to\infty}\left[\frac{-1}{2}e^{-x^2}\right]_0^t$ by (1)

$$= \lim_{t\to\infty}\left[\frac{-1}{2}e^{-t^2} + \frac{1}{2}\right] = \frac{1}{2}$$

$$\therefore (2) \Rightarrow \int_{-\infty}^{\infty} xe^{-x^2}dx = \frac{-1}{2} + \frac{1}{2} = 0$$

Example:

Evaluate $\int_3^\infty \frac{1}{(x-2)^{3/2}}dx$

Solution:

Consider $\int \frac{1}{(x-2)^{3/2}}dx \ldots(1)$

Put $u = x - 2 \quad \Rightarrow du = dx$

$$(1) \Rightarrow \int \frac{1}{(x-2)^{3/2}}dx = \int \frac{1}{u^{3/2}}du = \int u^{-3/2}\,du = \frac{u^{-3/2+1}}{-3/2+1} = \frac{u^{-1/2}}{-1/2}$$

$$\frac{-2}{\sqrt{u}} = \frac{-2}{\sqrt{x-2}}$$

$$\int_3^\infty \frac{1}{(x-2)^{3/2}}dx = \lim_{t\to\infty}\left[\int_3^t \frac{1}{(x-2)^{3/2}}dx\right] = \lim_{t\to\infty}\left[\frac{-2}{\sqrt{x-2}}\right]_3^t$$

$$= \lim_{t\to\infty}\left[\left(\frac{-2}{\sqrt{t-2}}\right) - \left(\frac{-2}{\sqrt{1}}\right)\right]$$

$$= \lim_{t\to\infty}\left(\frac{-2}{\sqrt{t-2}}\right) + 2 = 0 + 2 = 2 \text{ (finite)}$$

The given integral $\int_3^\infty \frac{1}{(x-2)^{3/2}}dx$ is convergent.

Example:

Evaluate $\int_0^2 \frac{1}{\sqrt{x}}dx$

Solution:

Here, infinite discontinuity occurs at x=0

$$\therefore \int_0^2 \frac{1}{\sqrt{x}}dx = \lim_{t\to 0^+}\int_t^2 x^{-1/2}dx$$

$$= \lim_{t\to 0^+}\left[\frac{x^{1/2}}{1/2}\right]_t^2$$

$$= \lim_{t\to 0^+}\left[2\sqrt{x}\right]_t^2$$

$$= \lim_{t \to 0^+} [2\sqrt{2} - 2\sqrt{t}]$$

$$= 2\sqrt{2} \text{ (finite)}$$

The given integral $\int_0^2 \frac{1}{\sqrt{x}} dx$ is convergent.

Example:

Evaluate $\int_0^3 \frac{1}{x-1} dx$

Solution:

Here, infinite discontinuity occurs at $x = 1$

$$\therefore \int_0^3 \frac{1}{x-1} dx = \int_0^1 \frac{1}{x-1} dx + \int_1^3 \frac{1}{x-1} dx$$

Take $\int_0^1 \frac{dx}{x-1} = \lim_{t \to 1^-} \int_0^t \frac{1}{x-1} dx = \lim_{t \to 1^-} [\log(x-1)]_0^t$

$$= \lim_{t \to 1^-} \log(t-1) = -\infty$$

$\int_0^1 \frac{1}{x-1} dx$ is divergent.

$\Rightarrow \int_1^3 \frac{1}{x-1} dx$ is also divergent.

The given integral $\int_0^3 \frac{1}{x-1} dx$ isdivergent

Example:

Evaluate $\int_2^5 \frac{1}{\sqrt{x-2}} dx$

Solution:

The infinite discontinuity occurs at $x = 2$

$$\therefore \int_2^5 \frac{1}{\sqrt{x-2}} dx = \lim_{t \to 2^+} \int_2^5 \frac{1}{\sqrt{x-2}} dx$$

$$= \lim_{t \to 2^+} [2\sqrt{x-2}]_t^5$$

$$= \lim_{t \to 2^+} (2\sqrt{3} - 2\sqrt{t-2})$$

$$= 2\sqrt{3} \text{ (finite)}$$

The given integral $\int_2^5 \frac{1}{\sqrt{x-2}} dx$ is convergent.

Example:

Evaluate $\int_0^3 \frac{1}{(x-1)^{2/3}} dx$

Solution:

Here infinite discontinuity occurs at $x = 1$

1) $\int_0^3 \frac{1}{(x-1)^{2/3}} dx = \int_0^1 \frac{1}{(x-1)^{2/3}} dx + \int_1^3 \frac{1}{(x-1)^{2/3}} dx \quad \ldots(1)$

Take $\int_0^1 \frac{1}{(x-1)^{2/3}} dx = \lim_{t\to 1^-} \int_0^t \frac{1}{(x-1)^{2/3}} dx$

$$= \lim_{t\to 1^-} [3(x-1)^{1/3}]_0^t$$

$$= \lim_{t\to 1^-} [3(t-1)^{1/3} + 3]$$

$$= 3$$

Take $\int_1^3 \frac{1}{(x-1)^{2/3}} dx = \lim_{t\to 1^+} \int_t^3 \frac{1}{(x-1)^{2/3}} dx$

$$= \lim_{t\to 1^+} [3(x-1)^{1/3}]_t^3$$

$$= \lim_{t\to 1^+} [3\,[2^{1/3} - (t-1)^{1/3}]]$$

$$= 3\,(2^{1/3})$$

(1) $\Rightarrow \int_0^3 \frac{1}{(x-1)^{2/3}} dx = 3 + 3\,(2^{1/3})$

$$= 3\ [1 + 2^{1/3}]$$

Comparison test for improper integrals

Let $\int_a^b f(x)dx$ be an improper integral.

i) If there exists a $g(x)$ such that $|f(x)| \le g(x)$ for all x in $[a,b]$ and $\int_a^b g(x)dx$ converges then $\int_a^b f(x)dx$ also converges.

ii) If there exists function $g(x)$ such that $f(x) \ge |g(x)|$ for all x in $[a,b]$ and $\int_a^b g(x)dx$ diverges then $\int_a^b f(x)dx$ also diverges.

Limit form of comparison Tests.

Let $f(x) > 0$ and $g(x) > 0$and $\lim_{x\to\infty} \frac{f(x)}{g(x)} = k$ where $k \neq 0$

Then, the improper integrals $\int_a^\infty f(x)dx$ and $\int_a^\infty g(x)dx$ converge or diverge together.

If $k = 0$, only the convergence of $\int_a^\infty g(x)dx$ implies that of $\int_a^\infty f(x)dx$

Absolute Convergence

The improper integral $\int_a^b f(x)dx$ is said to be absolutely convergent if $\int_a^b |f(x)|dx$ is convergent.

Note:

1) The same definition holds for $\int_a^\infty f(x)dx$ also
2) When the improper integral changes sign within the limits of the integration, then the above test is applied.

Example:

Discuss the convergence of $\int_1^\infty \frac{xtan^{-1}x}{\sqrt{4+x^3}} dx$

Solution:

Let $f(x) = \frac{xtan^{-1}x}{\sqrt{4+x^3}} = \frac{tan^{-1}x}{\sqrt{x}\sqrt{1+4x^{-3}}}$ and $g(x) = \frac{1}{\sqrt{x}}$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{tan^{-1}x}{\sqrt{1+4x^{-3}}}$$

$$= \frac{\pi}{2}$$

Hence, by comparision test 2, the integrals $\int_1^\infty f(x)dx$ and $\int_1^\infty g(x)dx$ converge or diverge together, Now $\int_1^\infty g(x)dx$ is divergent.

$\therefore \int_1^\infty f(x)dx$ is also divergent.

Example :

Discuss the convergence of $\int_1^\infty \frac{sinx}{x^4} dx$

Solution:

$$\left|\int_1^\infty \frac{sinx}{x^4}dx\right| \le \int_1^\infty \left|\frac{sinx}{x^4}\right| dx \le \int_1^\infty \frac{dx}{x^4}$$

$\Rightarrow$ convergent

$\int_1^\infty \frac{sinx}{x^4} dx$ is absolutely convergent and hence convergent.

Example:

Test the convergence of $\int_0^\infty e^{-x^2} dx$

Solution:

The given integral $\int_0^\infty e^{-x^2}\ dx$ is an improper integral of first kind and the integral can be written as $\int_0^\infty e^{-x^2}\ dx = \int_0^1 e^{-x^2}\ dx + \int_1^\infty e^{-x^2}\ dx$

The first integral in the right hand side $\int_0^1 e^{-x^2}\ dx$ is proper integral. So it is enough to check the second one.

We have that,

$$x \geq 1$$

$$x^2 \geq x$$

$$-x^2 \leq -x$$

$$e^{-x^2} \leq e^{-x}$$

$$\int_1^\infty e^{-x^2}\ dx \leq \int_1^\infty e^{-x}\ dx$$

$$= \lim_{b\to\infty} \int_1^b e^{-x}\ dx \qquad = \lim_{b\to\infty} [-e^{-x}]_1^b$$

$$= \lim_{b\to\infty} [e^{-1} - e^{-b}]$$

$$= [e^{-1} - 0] = \frac{1}{e}$$

Hence by comparison test the given integral is convergent.