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Limit of a function

Definition:

Suppose $f(x)$ is defined when x is near the number a, Then we write $\lim_{x \to a} f(x) = L$ and say the limit of $f(x)$, as x approaches a, equals L.

The above definition says that the value of $f(x)$ approach as x approaches a. In other words, the value of $f(x)$ tend to get closer and closer to L as x gets closer and closer to a from either side of a but $x \neq a$. The alternate notation for $\lim_{x \to a} f(x) = L$ is $f(x) \to L \text{ as } x \to a$.

One-sided Limits:

Left-hand limit of $f(x)$:

Suppose $f(x)$ is defined when x is near the number from left hand side of a, Then we write $\lim_{x \to a^-} f(x) = L$ and say the left-hand limit of $f(x)$, as x approaches a.

Right-hand limit of $f(x)$:

Suppose $f(x)$ is defined when x is near the number from right hand side of a, Then we write $\lim_{x \to a^+} f(x) = L$ and say the right-hand limit of $f(x)$, as x approaches a.

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Definition:

Suppose $f(x)$ is defined when x is near the number a. Then we write $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = L$.

Infinite Limits:

Suppose $f(x)$ is defined on both sides of 'a' except possibly at'a' itself. Then

(i) $\lim_{x \to a} f(x) = \infty$ means that the value of $f(x)$, can be made arbitrarily large by taking x to be sufficiently close to 'a' but not equal to a.

(ii) $\lim_{x \to a} f(x) = -\infty$ means that the value of $f(x)$, can be made arbitrarily large negative by taking x to be sufficiently close to 'a' but not equal to a.

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Example:

Evaluate $\lim_{x\to 2} x^2 - x + 2$

Solution:

Let $f(x) = x^2 - x + 2$

X	f(x)	x	f(x)
1.9	3.71	2.1	4.31
1.99	3.9701	2.01	4.0301
1.999	3.997001	2.001	4.003001
1.9999	3.99970001	2.0001	4.00030001
x<2		x>2	

From the table, $\lim_{x\to 2} x^2 - x + 2 = 4$

Example:

Find the value of $\lim_{x\to 1} \frac{x^2-1}{x-1}$

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Solution:

$f(x) = \frac{x^2-1}{x-1} = \frac{(x+1)(x-1)}{(x-1)}$

$= x + 1, x \neq 1$

X	f(x)	x	f(x)
0.9	1.9	1.1	2.1
0.99	1.99	1.01	2.01
0.999	1.999	1.001	2.001
$x < 1$		$x > 1$	

We can say f(x) approaches the limit 2 as x approaches 1.

$\therefore \lim_{x\to 1} \frac{x^2-1}{x-1} = 2$

Example:

Investigatelim $\lim_{x\to 0} sin \frac{\pi}{x}$

Solution:

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Let $f(x) = sin\frac{\pi}{x}$

x	1	1/3	0.1	1/2	1.4	0.01
f(x)	0	0	0	0	0	0

Our guess $\lim_{x\to 0} sin\frac{\pi}{x} = 0$ is wrong.

$\therefore f\left(\frac{1}{n}\right) = \sin n\pi = 0$ for any integer n.

$\therefore f\left(\frac{1}{n}\right) = 0$ which is not possible.

∴ The limit does not exists.

Example:

Use a table of values to estimate the value of the limit $\lim_{x\to 0} \frac{\sqrt{x+4}-2}{x}$

Solution:

Let $f(x) = \lim_{x\to 0} \frac{\sqrt{x+4}-2}{x}$

x	f(x)	x	f(x)
-1	0.2679	1	0.2361
-0.5	0.2583	0.5	0.2426
-0.1	0.2516	0.1	0.2485
-0.05	0.2508	0.05	0.2492
-0.01	0.2502	0.01	0.2498
-0.001	0.25	0.001	0.25

$$\therefore \lim_{x\to 0} \frac{\sqrt{x+4}-2}{x} = 0.25 = \frac{1}{4}$$

Example:

Evaluate the limit and justify each step for the following:

(i) $\lim_{x\to -1} (x^4 - 3x)(x^2 + 5x + 3)$

(ii) $\lim_{x\to -2} \frac{x^3+2x^2-1}{5-3x}$

(iii) $\lim_{u\to -2} \sqrt{u^4 + 3u + 6}$

Solution:

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(i) $\lim_{x\to -1}(x^4-3x)(x^2+5x+3) = \lim_{x\to -1}(x^4-3x)\lim_{x\to -1}(x^2+5x+3)$

$= [\lim_{x\to -1} x^4 - 3\lim_{x\to -1} x] \times [\lim_{x\to -1} x^2 + 5\lim_{x\to -1} x + \lim_{x\to -1} 3]$

$= [(-1)^4 - 3(-1)][(-1)^2 + 5(-1) + 3]$

$= 4(-1) = -4$

(ii) $\lim_{x\to -2}\frac{x^3+2x^2-1}{5-3x} = \frac{\lim_{x\to -2}(x^3+2x^2-1)}{\lim_{x\to -2}(5-3x)}$

$= \frac{(-2)^3+2(-2)^2-1}{5-3(-2)}$

$= \frac{-8+8-1}{5+6} = \frac{-1}{11}$

(iii) $\lim_{u\to -2}\sqrt{u^4+3u+6} = \sqrt{\lim_{u\to -2}(u^4+3u+6)}$

$= \sqrt{(-2)^4+3(-2)+6}$

$= \sqrt{16-6+6} = \sqrt{16} = 4$

Example:

Evaluate $\lim_{h\to 0}\frac{(3+h)^2-9}{h}$

Solution:

$\lim_{h\to 0}\frac{(3+h)^2-9}{h} = \lim_{h\to 0}\frac{9+6h+h^2-9}{h}$

$= \lim_{h\to 0}\frac{h(6+h)}{h}$

$= \lim_{h\to 0} 6+h = 6$

Example:

Evaluate $\lim_{x\to -4}\frac{x^2+5x+4}{x^2+3x-4}$

Solution:

$\lim_{x\to -4}\frac{x^2+5x+4}{x^2+3x-4} = \lim_{x\to -4}\frac{(x+1)(x+4)}{(x-1)(x+4)}$

$= \lim_{x\to -4}\frac{x+1}{x-1}$

$= \frac{-4+1}{-4-1} = \frac{-3}{-5} = \frac{3}{5}$

Example:

Evaluate the limit if it exists $\lim_{x\to 1}\frac{x^4-1}{x^3-1}$

Solution:

$\lim_{x\to 1}\frac{x^4-1}{x^3-1} = \lim_{x\to 1}\frac{(x-1)(x^3+x^2+x+1)}{(x-1)(x^2+x+1)}$

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$$= \lim_{x\to 1} \frac{(x^3+x^2+x+1)}{(x^2+x+1)}$$

$$= \frac{1+1+1+1}{1+1+1} = \frac{4}{3}$$

Example:

Evaluate $\lim_{t\to 0} \frac{\sqrt{1+t}-\sqrt{1-t}}{t}$

Solution:

$$\lim_{t\to 0} \frac{\sqrt{1+t}-\sqrt{1-t}}{t} = \lim_{t\to 0} \frac{\sqrt{1+t}-\sqrt{1-t}}{t} \times \frac{\sqrt{1+t}+\sqrt{1-t}}{\sqrt{1+t}+\sqrt{1-t}}$$

$$= \lim_{t\to 0} \frac{(\sqrt{1+t})^2-(\sqrt{1-t})^2}{t(\sqrt{1+t}+\sqrt{1-t})}$$

$$= \lim_{t\to 0} \frac{1+t-(1-t)}{t(\sqrt{1+t}+\sqrt{1-t})}$$

$$= \lim_{t\to 0} \frac{2t}{t(\sqrt{1+t}+\sqrt{1-t})}$$

$$= \frac{2}{\sqrt{1+0}+\sqrt{1-0}}$$

$$= \frac{2}{2} = 1$$

Example:

Evaluate $\lim_{x\to -4} \frac{\frac{1}{4}+\frac{1}{x}}{4+x}$

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Solution:

$$\text{Lim}_{x\to -4} \frac{\frac{1}{4}+\frac{1}{x}}{4+x} = \lim_{x\to -4} \frac{\frac{x+4}{4x}}{4+x}$$

$$= \lim_{x\to -4} \frac{1}{4x} = \frac{1}{-16}$$

Example:

Evaluate the limit if it exists $\lim_{x\to 2} \frac{x^2-x+6}{x-2}$

Solution:

$$\lim_{x\to 2} \frac{x^2-x+6}{x-2} = \frac{8}{0} = \infty$$

So the limit does not exists.

Example:

Evaluate the limit if it exists $\lim_{x\to -1} \frac{x^2-4x}{(x-4)(x+1)}$

Solution:

$$\lim_{x\to -1} \frac{x^2-4x}{(x-4)(x+1)} = \lim_{x\to -1} \frac{x(x-4)}{(x-4)(x+1)}$$

$$= \lim_{x\to -1} \frac{x}{(x+1)}$$

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$$= \frac{-1}{0} = \infty$$

∴ The limit does not exists.

Example:

Prove that $\lim_{x \to 0} |x| = 0$

Solution:

$$|x| = f(x) = \{ \begin{matrix} x, & x \geq 0 \\ -x, & x < 0 \end{matrix}$$

$$\lim_{x \to 0^+} |x| = \lim_{x \to 0^+} x = 0 \; for |x| = x \,, x > 0$$

$$\lim_{x \to 0^-} |x| = \lim_{x \to 0^-} (-x) = 0 \; for |x| = -x \,, x < 0$$

$$\therefore \lim_{x \to 0^+} |x| = 0 = \lim_{x \to 0^-} |x|$$

$$\lim_{x \to 0} |x| = 0$$

Example:

Prove that $\lim_{x \to 0} \frac{|x|}{x}$ *does not exist*.

Solution:

Let $f(x) = \frac{|x|}{x}$

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$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^+} \left(\frac{x}{x}\right) = \lim_{x \to 0^+} (1)$$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0^-} \left(\frac{-x}{x}\right) = \lim_{x \to 0^-} (-1) = -1$$

$$\lim_{x \to 0^+} f(x) \neq \lim_{x \to 0^-} f(x)$$

∴ $\lim_{x \to 0} \frac{|x|}{x}$ does not exist.

Example:

Let $g(x) = \frac{x^2+x-6}{|x-2|}$ *does* $\lim_{x \to 2} g(x)$ *exist*?

Solution:

$$\lim_{x \to 2^-} g(x) = \lim_{x \to 2^-} \frac{x^2+x-6}{-(x-2)}$$

$$= \lim_{x \to 2^-} \frac{(x-2)(x+3)}{-(x-2)}$$

$$= \lim_{x \to 2^-} -(x+3)$$

$$= -(2+3) = -5$$

$$\lim_{x \to 2^+} g(x) = \lim_{x \to 2^+} \frac{x^2+x-6}{(x-2)}$$

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$$= \lim_{x\to 2^+} \frac{(x-2)(x+3)}{(x-2)}$$

$$= \lim_{x\to 2^-} (2+3)$$

$$= 5$$

$$\lim_{x\to 2^-} g(x) \neq \lim_{x\to 2^+} g(x)$$

$\therefore \lim_{x\to 2} g(x)$ does not exist.

Example:

Find the limit if it exist $\lim_{x\to 0^-} \left(\frac{1}{x} - \frac{1}{|x|}\right)$

Solution:

$$\lim_{x\to 0^-} \left(\frac{1}{x} - \frac{1}{|x|}\right) = \lim_{x\to 0^-} \left(\frac{1}{x} - \frac{1}{-x}\right)$$

$$= \lim_{x\to 0^-} \left(\frac{1}{x} + \frac{1}{x}\right)$$

$$= \lim_{x\to 0^-} \left(\frac{2}{x}\right)$$

$$= \frac{2}{0} \quad = \infty$$

$\therefore$ Limit does not exist.

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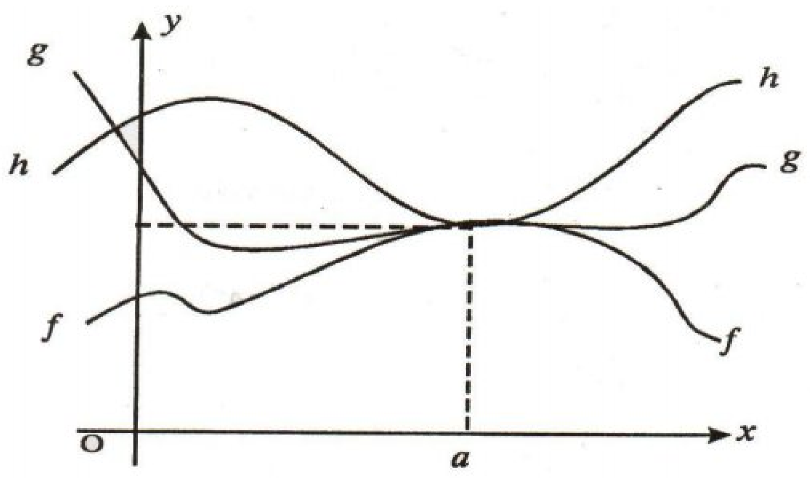
Squeeze theorem (or) Sandwich theorem (or) Pinching theorem:

Statement:

If $f(x) \leq g(x) \leq h(x)$ **when x is near a (except possibly at a) and**

$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$ then $\lim_{x\to a} g(x) = L$



ie , If g(x) is squeezed in between h(x) and f(x) which have the same limit L then g(x) also forced to have the same limit L.

Example:

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Show that $\lim_{x\to 0} x^2 sin\frac{1}{x} = 0$

Solution:

$$\lim_{x\to 0} x^2 sin\frac{1}{x} = \lim_{x\to 0} x^2 \lim_{x\to 0} sin\frac{1}{x}$$

Here $\lim_{x\to 0} sin\frac{1}{x}$ does not exists.

∴ By applying $x \to 0$ Squeeze theorem,

$$-1 \le sin\frac{1}{x} \le 1$$

$$-x^2 \le sin\frac{1}{x} \le x^2$$

$\lim_{x\to 0}(-x^2) = 0$ and $\lim_{x\to 0}(x^2) = 0$

By Squeeze theorem, $\lim_{x\to 0} x^2 sin\frac{1}{x} = 0$

Example:

Find $\lim_{\theta\to 0}\frac{tan\,\theta}{\theta}$

Solution:

$$\lim_{\theta\to 0}\frac{tan\,\theta}{\theta} = \lim_{\theta\to 0}\frac{sin\,\theta}{\theta cos\theta}$$

$$= \lim_{\theta\to 0}\left(\frac{sin\,\theta}{\theta}.\frac{1}{\cos\theta}\right)$$

$$= \lim_{\theta\to 0}\left(\frac{sin\,\theta}{\theta}\right).\lim_{\theta\to 0}\left(\frac{1}{\cos\theta}\right)$$

$$= 1.1 \quad = 1$$

Example:

Find $\lim_{\theta\to 0}\frac{1-cos\,x}{x}$

Solution:

$$\lim_{\theta\to 0}\frac{1-cos\,x}{x} = \lim_{\theta\to 0}\frac{2sin^2(\frac{x}{2})}{x}$$

$$= \lim_{\theta\to 0}\frac{sin^2(\frac{x}{2})}{(\frac{x}{2})} \text{ x } \frac{(x/2)}{(x/2)}$$

$$= \lim_{\theta\to 0}\left(\frac{sin(\frac{x}{2})}{(\frac{x}{2})}\right)^2 \text{ x } (\frac{x}{2})$$

$$= \lim_{\theta\to 0}\left(\frac{sin(\frac{x}{2})}{(\frac{x}{2})}\right)^2 \text{ x } \lim_{\theta\to 0}(\frac{x}{2})$$

$$= 1 \text{ x } 0 = 0$$

Example:

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Find $\lim_{x\to\frac{\pi}{2}} \frac{1+\cos 2x}{(\pi-2x)^2}$

Solution:

$$\lim_{x\to\frac{\pi}{2}} \frac{1+\cos 2x}{(\pi-2x)^2} = \lim_{x\to\frac{\pi}{2}} \frac{2\cos^2 x}{(\pi-2x)^2}$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{2\sin^2(\frac{\pi}{2}-x)}{2^2(\frac{\pi}{2}-x)^2}$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{1}{2}\left[\frac{\sin(\frac{\pi}{2}-x)}{(\frac{\pi}{2}-x)}\right]^2$$

$$= \frac{1}{2} \lim_{(x-\frac{\pi}{2})\to 0} \left[\frac{\sin(-)(x-\frac{\pi}{2})}{-(x-\frac{\pi}{2})}\right]^2 = \frac{1}{2}(1) = \frac{1}{2}$$

Example:

Find $\lim_{x\to 0} \frac{\sin^2(\frac{x}{3})}{x^2}$

Solution:

$$\lim_{x\to 0} \frac{\sin^2(\frac{x}{3})}{x^2} = \lim_{x\to 0} \frac{\sin^2(\frac{x}{3})}{x^2(\frac{1}{3^2})} \text{x} \left(\frac{1}{3^2}\right)$$

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$$= \lim_{x\to 0} \frac{\sin^2(\frac{x}{3})}{(\frac{x}{3})^2} \text{x} \left(\frac{1}{3^2}\right)$$

$$= \lim_{x\to 0} \left[\frac{\sin(\frac{x}{3})}{(\frac{x}{3})}\right]^2 \text{x} \lim_{x\to 0} \left(\frac{1}{9}\right)$$

$$= 1 \text{ x } \frac{1}{9} = \frac{1}{9}$$

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Representation of functions

Functions:

A function is a rule that assigns to each element x in a set A to exactly one element called $f(x)$ in a set B.

Odd and Even functions:

If a function f satisfies $f(-x) = f(x)$ for every number x in its domain, then f is called an even function. **Example:** $cosx$, x^2 , x^4, $|x|$ are even functions.

If a function f satisfies $f(-x) = -f(x)$ for every number x in its domain, then f is called an odd function. **Example:** $sin\ x$, x , x^3 are odd functions.

Graph of functions:

If f is a function with domain D, then its graph is the set of ordered pair $\{(x, f(x))/x\epsilon D\}$.

Domain, Co-domain, Range and Image:

Let: $A \to B$, then the set A is called the domain of the function and set B is called Co-domain.

The set of all the images of all the elements of A under the function f is called the range of f and it is denoted by $f(A)$.

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Range of f is $f(A) = \{ f(x): x\epsilon A \}$

clearly $f(A) \subseteq B$

If $x \in A$, $y \in B$ and $y = f(x)$ then y is called the image of x under f.

Find the domain and range of the function:

(i) $f(x) = \frac{1}{x^2-x}$ **(ii)** $f(x) = \frac{4}{3-x}$ **(iii)** $f(x) = \sqrt{5x+10}$ **(iv)** $f(x) = 1 + x^2$ **(v)** $f(x) = \sqrt{x+2}$

(i) $f(x) = \frac{1}{x^2-x}$

Solution:

$$x^2 - x = 0 \Longrightarrow x(x-1) = 0$$

$$\Longrightarrow x = 0 , x - 1 = 0 \Rightarrow x = 1$$

Domain is $(-\infty, 0) \cup (0,1) \cup (1, \infty)$

Range is $(0, \infty)$

(ii) $f(x) = \frac{4}{3-x}$

Solution:

$$3 - x = 0 \Longrightarrow x = 3$$

Domain is $(-\infty, 3) \cup (3, \infty)$

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Range is $(-\infty, 0) \cup (0, \infty)$

(iii) $f(x) = \sqrt{5x + 10}$

Solution:

Since square root of a negative number is not defined, $5x + 10 \geq 0$

$$\Longrightarrow 5x \geq -10 \Longrightarrow x \geq -2$$

Domain is $[-2, \infty)$

Range is $[0, \infty)$

(iv) $f(x) = 1 + x^2$

Solution:

ie, $y = 1 + x^2 \Rightarrow y - 1 = x^2$

Here $x^2 \geq 0 \Longrightarrow y - 1 \geq 0 \Longrightarrow y \geq 1$

Domain is $[-\infty, \infty)$

Range is $[1, \infty)$

(v) $f(x) = \sqrt{x + 2}$

Solution:

Since square root of a negative number is not defined, $x + 2 \geq 0 \Longrightarrow x \geq -2$

Domain is $[-2, \infty)$

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Range is $[0, \infty)$

Find the domain and sketch the graph of the function $f(x) = \begin{cases} x + 2 & if\ x < 0 \\ 1 - x & if\ x \geq 0 \end{cases}$

Solution:

Given $f(x) = \begin{cases} x + 2 & if\ x < 0 \\ 1 - x & if\ x \geq 0 \end{cases}$

ie, $y = x + 2,\ x < 0 \qquad y = 1 - x, x \geq 0$

x<0	-1	-2	-3	-4	…
y = x+2	1	0	-1	-2	…

x ≥0	0	1	2	3	…
y = 1-x	1	0	-1	-2	…

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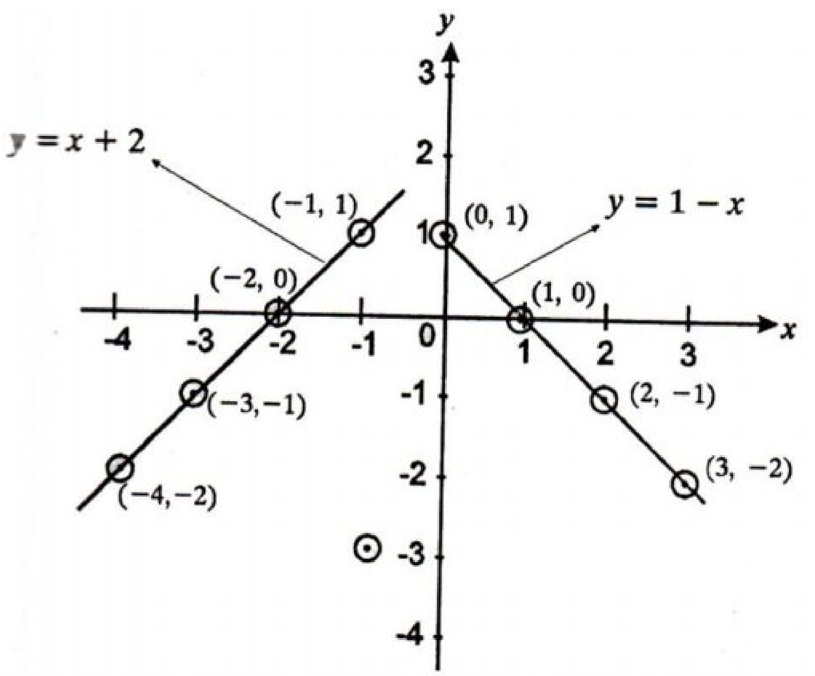

Domain is $(-\infty, \infty)$

Example:

Sketch the graph of the absolute value function $f(x) = |x|$

Solution:

Let $y = f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

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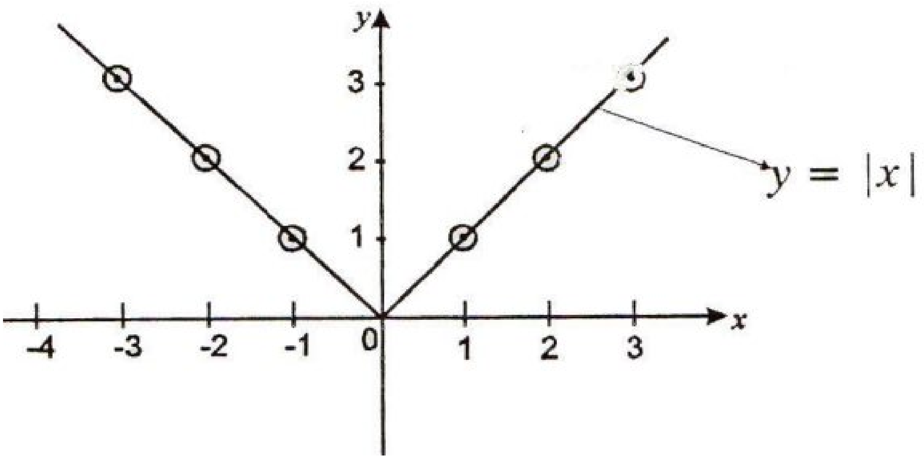

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Continuity

A function f is continuous at a number 'a' if $\lim_{x\to a} f(x) = f(a)$

Note: (i)

If f is continuous at a, then

1. $f(a)$ should exist.
2. $\lim_{x\to a} f(x)$ exist both on the left and right
3. $\lim_{x\to a} f(x) = f(a)$

The definition says that f is continuous of a if $f(x)$ approaches $f(a)$ as x approaches a.

Note: (ii)

The function $f(x)$ is said to be discontinuous at $x = a$ if one or more of the above three conditions are not satisfied.

Example:

How would you remove the discontinuity of $f(x) = \frac{x^3-8}{x^2-4}$

Solution:

$\lim_{x\to a} f(x) = f(a)$

Given $f(x) = \frac{x^3-8}{x^2-4}$

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$f(x)$ is defined in all the real vlues except at $x = 2$.

$\therefore f(2)$ is not defined.

$$\text{But} \lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x^3-8}{x^2-4} = \lim_{x\to 2} \frac{(x-2)(x^2+2x+4)}{(x+2)(x-2)}$$

$$= \lim_{x\to 2} \frac{(x-2)(x^2+2x+4)}{(x+2)(x-2)}$$

$$= \lim_{x\to 2} \frac{(x^2+2x+4)}{(x+2)}$$

$$= \frac{4+4+4}{2+2}$$

$$= \frac{12}{4} = 3$$

Then the discontinuity is removed.

$$\therefore \text{The function is defined as} \begin{cases} \frac{x^3-8}{x^2-4} & if\ x \neq 2 \\ 3 & if\ x = 2 \end{cases}$$

Example.

Discuss the continuity of the function $\frac{x^2-x-2}{x-2}$

Solution:

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A function f is continuous at 'a' if $\lim_{x\to a} f(x) = f(a)$

The given function $\frac{x^2-x-2}{x-2}$ is defined for all real value of x except at $x = 2$.

So $f(2)$ is not defined.

Hence the function is discontinuous at $x = 2$.

Example:

Evaluate $\lim_{x\to 1} \sin^{-1}\left(\frac{1-\sqrt{x}}{1-x}\right)$ **(or)** $\lim_{x\to 1} arc \sin \frac{1-\sqrt{x}}{1-x}$

Solution:

$$\lim_{x\to 1} \sin^{-1}\left(\frac{1-\sqrt{x}}{1-x}\right) = \lim_{x\to 1} \sin^{-1}\left(\frac{1-\sqrt{x}}{1-(\sqrt{x})^2}\right)$$

$$= \lim_{x\to 1} \sin^{-1}\left(\frac{1-\sqrt{x}}{(1+\sqrt{x})(1-\sqrt{x})}\right)$$

$$= \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

Example:

Show that the junction $f(x) = 1 - \sqrt{1-x^2}$ is continuous on the interval [-1, 1]

Solution:

Given $f(x) = 1 - \sqrt{1-x^2}$ in [-1, 1]

Let $a \in [-1, 1]$, i.e., $-1 < a < 1$

To Prove $\lim_{x\to a} f(x) = f(a)$

L.H.S = $\lim_{x\to a} f(x)$

$= \lim_{x\to a}[1-\sqrt{1-x^2}]$

$= 1 - \sqrt{1-a^2}$

$= f(a)$

∴ The given function is continuous.

Example:

For what value of the constant b is the function f continuous on (-∞, ∞)

$$f(x) = \begin{cases} bx^2 + 2x & if \quad x < 2 \\ x^3 - bx & if \quad x \geq 2 \end{cases}$$

Solution:

Given the function is continuous.

∴ $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x)$

$\lim_{x\to 2} (bx^2 + 2x) = \lim_{x\to 2}(x^3 - bx)$

$\Rightarrow 4b + 4 = 8 - 2b$

$\Rightarrow 4b + 2b = 8 - 4$

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$\Rightarrow 6b = 4$

$\Rightarrow b = \frac{4}{6} = \frac{2}{3}$

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Maxima and Minima of functions of one variable

Let c be a point in a domain D of a function f. Then f(c) is the

⇒ absolute maximum value of f on D if $f(c) \geq f(x)$ for all x in D.

⇒ absolute minimum value of f on D if $f(c) \leq f(x)$ for all x in D.

Definition:

Let c be a point in a domain D of a function f. Then f(c) is the

⇒ local maximum value of f if $f(c) \geq f(x)$ when x is near c.

⇒ local minimum value of f if $f(c) \leq f(x)$ when x is near c.

Critical Point:

A critical point of a function f is a point c in the domain of f such that either f '(c) = 0 or f '(c) does not exists.

If f has local maximum value or minimum value at c, then c is a critical point of f.

Example:

Find the critical points of the following functions

(i) $\boldsymbol{f(x) = x^3 + x^2 - x}$

(ii) $\boldsymbol{f(x) = x^{\frac{5}{4}} - 2x^{\frac{1}{4}}}$

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Solutions:

(i) $\boldsymbol{f(x) = x^3 + x^2 - x}$

$$f'(x) = 3x^2 + 2x - 1$$

$$f'(x) = 0 \Rightarrow 3x^2 + 2x - 1 = 0$$

$$\Rightarrow (3x - 1)(x + 1) = 0$$

$$\Rightarrow x = \frac{1}{3}\ , -1$$

Critical points are $x = \frac{1}{3}\ , -1$.

(ii) $\boldsymbol{f(x) = x^{\frac{5}{4}} - 2x^{\frac{1}{4}}}$

$$f'(x) = \frac{5}{4}x^{\frac{1}{4}} - \frac{1}{4}2x^{-\frac{3}{4}}$$

$$f'(x) = 0 \Rightarrow \frac{1}{4}x^{\frac{1}{4}}(5 - 2x^{-1}) = 0$$

$$\Rightarrow \frac{1}{4}x^{\frac{1}{4}} = 0\ ,\ (5 - 2x^{-1}) = 0$$

$$\Rightarrow x = 0\ ,\ \frac{5x-2}{x} = 0$$

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$\Rightarrow x = \frac{2}{5}$

Critical points are $x = 0, \frac{2}{5}$.

Example:

Find the absolute maximum and absolute minimum of

(i) $f(x) = 3x^4 - 4x^3 - 12x^2 + 1$ on [-2 , 3]

(ii) $f(x) = x - 2sinx$ on $[0, 2\pi]$

(iii) $(x) = x - logx$ on $[\frac{1}{2}, 2]$

Solutions:

(i) $f(x) = 3x^4 - 4x^3 - 12x^2 + 1$

$f(x) = 3x^4 - 4x^3 - 12x^2 + 1$ is continuous on [-2 , 3]

$f'(x) = 12x^3 - 12x^2 - 24x$

$f'(x) = 0 \Rightarrow 12x^3 - 12x^2 - 24x = 0$

$\Rightarrow x(x+1)(x-2) = 0$

$\Rightarrow x = 0, -1, 2$ are the critical points.

The values of $f(x)$ at critical points are

$f(0) = 3(0^4) - 4(0^3) - 12(0^2) + 1 = 1$

$f(-1) = 3(-1)^4 - 4(-1)^3 - 12(-1)^2 + 1$

$= 3 + 4 - 12 + 1 = -4$

$f(2) = 3(2)^4 - 4(2)^3 - 12(2)^2 + 1$

$= 48 - 32 - 48 + 1 = -31$

The value of $f(x)$ at the end points of the interval are

$f(-2) = 3(-2)^4 - 4(-2)^3 - 12(-2)^2 + 1$

$= 48 + 32 - 48 + 1 = 33$

$f(3) = 3(3)^4 - 4(3)^3 - 12(3)^2 + 1$

$= 243 - 112 - 108 + 1 = 28$

Absolute minimum value is $f(2) = -31$

Absolute maximum value is $f(-2) = 33$

(ii) f(iv) $f(x) = x - 2sinx$ on $[0, 2\pi]$

Solution:

$f(x) = x - 2sinx$ is continuous on $[0, 2\pi]$

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$f'(x) = 1 - 2cosx$

$f'(x) = 0 \Rightarrow 1 - 2cosx = 0$

$\Rightarrow cosx = \frac{1}{2}$

$\Rightarrow x = cos^{-1}\left(\frac{1}{2}\right)$

$\Rightarrow x = \frac{\pi}{3}, \frac{5\pi}{3}$ are the critical points.

The values of f(x) at critical points are

$f\left(\frac{\pi}{3}\right) = \frac{\pi}{3} - 2sin\frac{\pi}{3}$

$= \frac{\pi}{3} - 2\frac{\sqrt{3}}{2}$

$= \frac{\pi}{3} - \sqrt{3} \approx 0.684853$

$f\left(\frac{5\pi}{3}\right) = \frac{5\pi}{3} - 2sin\frac{5\pi}{3}$

$= \frac{5\pi}{3} - 2\left(-\frac{\sqrt{3}}{2}\right)$

$= \frac{5\pi}{3} + \sqrt{3} \approx 6.968039$

The values of f(x) at the end points of the intervals are

$f(0) = 0 - 2sin0 = 0$

$f(2\pi) = 2\pi - 2\sin(2\pi) = 2\pi = 6.28$

Absolute minimum value is $f\left(\frac{\pi}{3}\right) = -0.684$

Absolute maximum value is $f\left(\frac{5\pi}{3}\right) = 6.9680$

(iii) $f(x) = x - logx$ on $[\frac{1}{2}, 2]$

Solution:

$f(x) = x - logx$ is continuous on $[\frac{1}{2}, 2]$

$f'(x) = 1 - \frac{1}{x}$

$f'(x) = 0 \Rightarrow 1 - \frac{1}{x} = 0$

$\Rightarrow \frac{x-1}{x} = 0$

$\Rightarrow x = 1$ is the critical point.

The value of f(x) at critical point is

$f(1) = 1 - log1 = 1 - 0 = 1$

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The values of f(x) at the end points of the intervals are

$$f\left(\frac{1}{2}\right) = \frac{1}{2} - log\,\frac{1}{2}$$

$$= \frac{1}{2} - (-0.6931)$$

$$= 1.1931$$

$$f(2) = 2 - \log 2$$

$$= 2 - 0.6931$$

$$= 1.3068$$

Absolute maximum value is f(2) = 1.3068

Absolute minimum value is f(1) = 1

Rolle's Theorem:

Let f be a function that satisfies the following three conditions:

1) f is continuous on the closed interval $[a, b]$

2) f is differentiable on the open interval (a, b)

3) $f(a) = f(b)$

Then there exists a number c in (a, b) such that $f\,'(c) = 0$

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Example:

Verify Rolle's theorem for the following functions on the given interval

a) $\boldsymbol{f(x) = x^3 - x^2 + 6x + 2\,, [0, 3]}$

b) $\boldsymbol{f(x) = \sqrt{x} - \frac{1}{3}x\,, [0, 9]}$

Solutions:

a) $\boldsymbol{f(x) = x^3 - x^2 + 6x + 2\,, [0, 3]}$

Solution:

$f(x)$ is continuous on $[0, 3]$

$f(x)$ is differentiable on $[0, 3]$

$$f(0) = 2$$

$$f(3) = 27 - 9 + 18 + 2 = 38$$

$$f(0) \neq f(3)$$

Hence the Rolle's theorem is not satisfied.

b) $\boldsymbol{f(x) = \sqrt{x} - \frac{1}{3}x\,, [0, 9]}$

Solution:

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$f(x)$ is continuous on $[0, 9]$

$f(x)$ is differentiable on $[0, 9]$

$f(0) = 0$

$f(9) = \sqrt{9} - \frac{9}{3} = 3 - 3 = 0$

$f(0) = 0 = f(9)$

$\Rightarrow f(x) = \sqrt{x} - \frac{x}{3}$

$\Rightarrow \ f'(x) = \frac{1}{2\sqrt{x}} - \frac{1}{3}$

$\Rightarrow f'(x) = 0 \Rightarrow \frac{1}{2\sqrt{x}} - \frac{1}{3} = 0$

$\Rightarrow \frac{1}{2\sqrt{x}} = \frac{1}{3}$

$\Rightarrow \sqrt{x} = \frac{3}{2}$

Squaring, $x = \frac{9}{4} = 2.25 \in (0,9)$

Hence Rolle's theorem is verified.

Example:

Prove that equation $x^3 - 15x + c = 0$ has atmost one real root in the interval [-2 , 2]

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Solution:

$Let\ f(x) = x^3 - 15x + c = 0$

$f(-2) = -8 + 30 + c = 22 + c$

$f(2) = 8 - 30 + c = -22 + c$

$f'(x) = 3x^2 - 15$

Now if there were two points x = a, b such that $f(x) = 0$

∴By Rolle's theorem there exists a point $x = c$ in between them, where $f'(c) = 0$

Now $f'(x) = 0 \Rightarrow 3x^2 - 15 = 0$

$\Rightarrow x^2 = 5$

$\Rightarrow x = \pm\sqrt{5} = \pm 2.236$

Here both values lies outside [-2, 2]

∴ f has no more than one zero.

$\Rightarrow f(x)$ has exactly one real root.

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Example:

Let $f(x) = 1 - x^{2/3}$, Show that $f(-1) = f(1)$but there is no number c in (-1 , 1) such that $f'(x) = 0$. Why does this not contradict Rolle's theorem?

Solution:

Given $f(x) = 1 - x^{2/3}$

$\Rightarrow f(-1) = 1 - (-1)^{\frac{2}{3}} = 0$

$\Rightarrow f(1) = 1 - 1^{2/3} = 0$

$\therefore f(-1) = f(1)$

$\Rightarrow f'(x) = -\frac{2}{3}x^{-1/3}$

$\Rightarrow f'(x) = 0 \Rightarrow -\frac{2}{3}x^{-1/3} = 0$

$\Rightarrow x^{-1/3} = 0$

$\Rightarrow (x^{-1/3})^3 = 0^3$

$\Rightarrow x^{-1} = 0$

$\Rightarrow \frac{1}{x} = 0$

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$\Rightarrow x = \infty$

There is no number c in $(-1 , 1)$

f is not differentiable on $(-1 , 1)$

Increasing/ Decreasing Test

Definition:

(a) If $f'(x) > 0$ on an interval, then f is increasing on that interval.

(b) If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

The first derivative test

Definition:

Suppose that c is a critical number of a continuous function f.

(a) If f ' changes from positive to negative at c , then f has a local maximum at c.

(b) If f ' changes from negative to positive at c , then f has a local minimum at c.

(c) If f' does not change sign at c (for example if f' is positive on both sides of c or negative on both sides), then f has no local maximum or minimum at c.

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Definition:

If the graph of f lies above all of its tangents on an interval I, then it is called concave upward on I. If the graph of f lies below all of its tangents on an interval I, then it is called concave downward on I.

Note:

Concave upward $\equiv$ convex downward

Concave downward $\equiv$ convex upward

Concavity Test

Definition:

If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.

(a) If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

Definition:

A point P on a curve $y = f(x)$is called an inflection point iff is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at P.

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The Second Derivative Test

Definition:

Suppose f'' is continuous near c,

(a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.

(b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

Example:

Find where the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing and where it is decreasing.

Solution:

Given $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$

$$f'(x) = 12x^3 - 12x^2 - 24x$$

$$= 12x(x^2 - x - 2)$$

$$= 12x(x - 2)(x + 1)$$

$$f'(x) = 0 \Rightarrow 12x(x - 2)(x + 1) = 0$$

$$\Rightarrow x(x - 2)(x + 1) = 0$$

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$\Rightarrow x = 0, 2, -1$ are the critical values.

We divide the real line into intervals whose end points are the critical points. $x = 0, 2, -1$ and list them in a table

Interval	$12x$	$x-2$	$x+1$	$f'(x)$	$f(x)$
$x < -1$	-	-	-	-	decreasing
$-1 < x < 0$	-	-	+	+	increasing
$0 < x < 2$	+	-	+	-	decreasing
$x > 2$	+	+	+	+	increasing

∴The function is increasing in $-1 < x < 0$ and $x > 2$ and it is decreasing in $x < -1$ and $0 < x < 2$

Example:

Find the local maximum and minimum values of $y = x^5 - 5x + 3$ using both the first and second derivative tests.

Solution:

Given $y = f(x) = x^5 - 5x + 3$

$f'(x) = 5x^4 - 5$

$f'(x) = 0 \Rightarrow 5x^4 - 5 = 0$

$\Rightarrow x^4 - 1 = 0 \Rightarrow x^4 = 1 \Rightarrow x^2 = \pm 1$

$\Rightarrow x = 1, -1$ are the critical points.

Interval	Sign of f'	Behaviour of f
$-\infty < x < 1$	+	increasing
$-1 < x < 1$	-	decreasing
$1 < x < \infty$	+	increasing

First derivative test tells us that

(i) Local maximum at $x = -1$

$f(-1) = -1 + 5 + 3 = 7$

Second derivative test tells us that

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(ii) Local minimum at $x = 1$

$f(1) = 1 - 5 + 3 = -1$

$f''(x) = 20x^3$

$f''(x) = 0 \Rightarrow 20x^3 = 0 \Rightarrow x = 0$

Interval	$f''(x)$	Behaviour of f
(-∞,0)	-	Concave down
(0,∞)	+	Concave up

$f'(1) = 0$, $f''(1) = 20$, $f(1) = -1$ is a local minimum

$f'(-1) = 0$, $f''(-1) = -20$, $f(-1) = 7$ is a local maximum

Example:

If $f(x) = 2x^3 + 3x^2 - 36x$ find the intervals on which is increasing or decreasing, the local maximum and local minimum values of f , the intervals of concavity and the inflection points.

Solution:

Given $f(x) = 2x^3 + 3x^2 - 36x$

$f'(x) = 6x^2 + 6x - 36$

$f'(x) = 0 \Rightarrow 6(x^2 + x - 6) = 0$

$\Rightarrow 6(x + 3)(x - 2) = 0$

$\Rightarrow x = -3$, 2are the critical points.

$f''(x) = 12x + 6$

We divide the real line into intervals whose end points are the critical points $x = 2, -3$ and list them in a table.

Interval	$6(x + 3)$	$x - 2$	$f'(x)$	$f(x)$
$x < -3$	$-$	$-$	$+$	increasing
$-3 < x < 2$	$+$	$-$	$-$	decreasing
$x > 2$	$+$	$+$	$+$	increasing

Now we apply the first derivative test to find the local extremum values.

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$f(x)$changes from increasing to decreasing at $x = -3$. Thus the function has a local maximum

$x = -3$ and local maximum value is $f(-3) = 2(-3)^3 + 3(-3)^2 - 36(-3)$

$= 2(-27) + 3(9) + 108$

$= -54 + 27 + 108 \quad = 81$

$f(x)$ changes from decreasing to increasing at $x = 2$. Thus the function has a local minimum

$x = 2$ and local minimum value is $f(2) = 2(2)^3 + 3(2)^2 - 36(2)$

$= 2(8) + 3(4) - 72$

$= 16 + 12 - 7 = -44$

For concavity test ,$f''(x) = 0$

$\Rightarrow 12x + 6 = 0$

$\Rightarrow x = -\frac{1}{2}$

We divide the real line into intervals whose end points are the critical points $x = -\frac{1}{2}$and list them in a table.

Interval	$f''(x)$	concavity
$x < -1/2$	$-$	downward
$x > -1/2$	$+$	upward

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Since the curve changes from concave downward to concave upward at $x = -\frac{1}{2}$

The point of inflection is $[-\frac{1}{2}, f(-\frac{1}{2})]$

$$f(-\frac{1}{2}) = 2(-\frac{1}{2})^3 + 3(-\frac{1}{2})^2 - 36(-\frac{1}{2})$$

$$= 2(-\frac{1}{8}) + 3(\frac{1}{4}) + 18$$

$$= -\frac{1}{4} + \frac{3}{4} + 18$$

$$= \frac{-1+3+72}{4}$$

$$= \frac{74}{4} = \frac{37}{2}$$

Hence the point of inflection are $(-\frac{1}{2}, \frac{37}{2})$

Example:

Find the interval of concavity and the inflection points. Also find the extreme values on what interval is f increasing or decreasing.

a) $(x) = sinx + cosx\ , 0 \leq x \leq 2\pi$

b) $f(x) = e^{2x} + e^{-x}$

c) $f(x) = x + 2sinx\ , 0 \leq x \leq 2\pi$

Solution:

a) $(x) = sinx + cosx\ , 0 \leq x \leq 2\pi$

$f'(x) = cosx - sinx$

$f'(x) = 0 \Rightarrow cosx = sinx$

$\Rightarrow x = \frac{\pi}{4}, \frac{5\pi}{4}$ are the critical points.

Interval	Sign of f '	Behaviour of f
$0 < x < \frac{\pi}{4}$	+	increasing
$\frac{\pi}{4} < x < \frac{5\pi}{4}$	+	increasing
$\frac{5\pi}{4} < x < 2\pi$	-	decreasing

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(i) Maximum at $\frac{\pi}{4}$, $f'(\frac{\pi}{4}) = sin\frac{\pi}{4} + cos\frac{\pi}{4}$

$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$

(ii) Minimum at $\frac{5\pi}{4}$, $f'(\frac{5\pi}{4}) = sin\frac{5\pi}{4} + cos\frac{5\pi}{4}$

$= -\sqrt{2}$

$f''(x) = -sinx - cosx = -(sinx + cosx)$

$f''(x) = 0 \Rightarrow -(sinx + cosx) = 0$

$\Rightarrow sinx = -cosx$

$\Rightarrow x = \frac{3\pi}{4}, \frac{7\pi}{4}$

Interval	Sign of f''	Behaviour of f

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$0 < x < \frac{3\pi}{4}$	$-$	Concave down
$\frac{3\pi}{4} < x < \frac{7\pi}{4}$	$+$	Concave up
$\frac{3\pi}{4} < x < 2\pi$	$-$	Concave down

Inflection points are $(\frac{3\pi}{4}, 0)$, $(\frac{7\pi}{4}, 0)$

Since $f\,(\frac{3\pi}{4}) = 0, f\,(\frac{7\pi}{4}) = 0$

b) $f(x) = e^{2x} + e^{-x}$

$f'(x) = 2e^{2x} - e^{-x}$

$f'(x) = 0 \Rightarrow 2e^{2x} - e^{-x} = 0$

$\Rightarrow 2e^{2x} = e^{-x}$

$\Rightarrow e^{3x} = \frac{1}{2}$

$\Rightarrow 3x = \log(\frac{1}{2})$

$\Rightarrow x = \frac{1}{3}[\log 1 - log2]$

$\Rightarrow x = \frac{1}{3}[0 - 0.693]$

$\Rightarrow -0.23$are the critical points.

Interval	Sign of f '	Behaviour of f
$-\infty < x < -0.23$	$-$	decreasing
$-0.23 < x < \infty$	$+$	increasing

The first derivative test tells us that there is a local minimum at $x = -0.23$

$f(-0.23) = f\,(-\frac{1}{3}log2) = f\,(log2^{-\frac{1}{3}})$

$= e^{2log2^{-1/3}} + e^{-log2^{-1/3}}$

$= e^{log(2^{-1/3})^2} + e^{log(2^{-1/3})^{-1}}$

$= (2^{-1/3})^2 + (2^{-1/3})^{-1}$

$= (2)^{-2/3} + (2)^{1/3}$

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$f''(x) = 4e^{2x} + e^{-x}$

$f''(x) = 0 \Rightarrow 4e^{2x} + e^{-x} = 0$

$\Rightarrow 4e^{2x} = -e^{-x}$

$\Rightarrow e^{3x} = -\frac{1}{4}$

$\Rightarrow 3x = log\,(-\frac{1}{4})$

$\Rightarrow x = \frac{1}{3} log\,(-\frac{1}{4})$

$\Rightarrow x = \frac{1}{3}(-log\,4\,)$

$= -\frac{1}{3}(log\,4\,) = -0.46$

Interval	Sign of f''	Behaviour of f
$-\infty < x < -0.46$	+	Concave up
$-0.46 < x < \infty$	+	Concave up

No inflection points.

c) $f(x) = x + 2sinx\,,\ 0 \le x \le 2\pi$

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$f'(x) = 1 + 2cosx$

$f'(x) = 0 \Rightarrow 2cosx = -1$

$\Rightarrow cosx = -\frac{1}{2}$

$\Rightarrow x = \frac{2\pi}{3}, \frac{4\pi}{3}$ are the critical points.

Interval	Sign of f'	Behaviour of f
$0 < x < \frac{2\pi}{3}$	+	increasing
$\frac{2\pi}{3} < x < \frac{4\pi}{3}$	−	decreasing
$\frac{4\pi}{3} < x < 2\pi$	+	increasing

The first derivatives test tells us that there is a

(i) Local maximum at $\frac{2\pi}{3}$

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$$f\left(\frac{2\pi}{3}\right) = \frac{2\pi}{3} + 2\sin\left(\frac{2\pi}{3}\right) = 3.83$$

(ii) Local minimum at $\frac{4\pi}{3}$

$$f\left(\frac{4\pi}{3}\right) = \frac{4\pi}{3} + 2\sin\left(\frac{4\pi}{3}\right) = 2.46$$

$$f''(x) = -2sinx$$

$$f''(x) = 0 \Rightarrow -2sinx = 0$$

$$\Rightarrow sinx = 0 \Rightarrow x = 0, \pi, 2\pi$$

Interval	Sign of f''	Behaviour of f
$0 < x < \pi$	$+$	Concave up
$\pi < x < 2\pi$	$-$	Concave down

Inflection points are (π, π)

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