Catalog

UNIT-V
LINEAR TIME INVARIANT DISCRETE TIME SYSTEMS

5.1 INTRODUCTION

A discrete-time system is anything that takes a discrete-time signal as input and generates a discrete-time signal as output.1 The concept of a system is very general. It may be used to model the response of an audio equalizer . In electrical engineering, continuous-time signals are usually processed by electrical circuits described by differential equations.

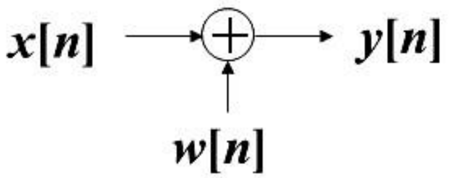
For example, any circuit of resistors, capacitors and inductors can be analyzed using mesh analysis to yield a system of differential equations. The voltages and currents in the circuit may then be computed by solving the equations. The processing of discrete-time signals is performed by discrete-time systems. Similar to the continuous-time case, we may represent a discrete-time system either by a set of difference equations or by a block diagram of its implementation.

For example, consider the following difference equation. $y(n) = y(n-1)+x(n)+x(n-1)+x(n-2)$ This equation represents a discrete-time system. It operates on the input signal x(n) to produce the output signal y(n).

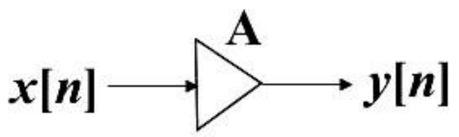
5.2 BLOCK DIAGRAM REPRESENTATION

A structural representation using interconnected basic building blocks is the first step in hardware or software implementation of an LTI system.

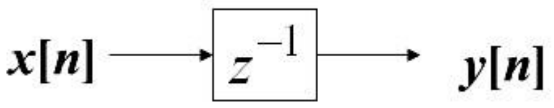
Basic Building Blocks:



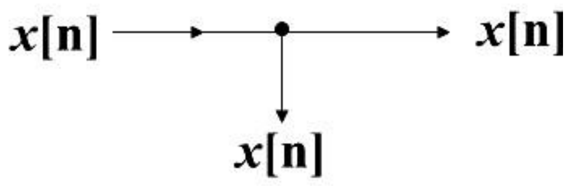
Adder



Multiplier



Unit delay



Pick-off node

There are four types of system realization in continuous time linear time invariant systems. They are

- Direct form I realization
- Direct form II realization
- Cascade form realization
- Parallel form realization

DIRECT FORM I REALIZATION

It is the direct implementation of differential equation or transfer function describing the system. It uses separate integrators for input and output variables. It provides direct relation between time domain and s-domain equations. In general, this form requires 2N delay elements (for both input and output signals) for a filter of order N. This form is practical for small filters.

Advantages:

- Simplicity
- Most straight forward realization

Disadvantages:

- More number of integrators are used
- Inefficient and impractical (numerically unstable) for complex design

The transfer function H(z) of the IIR system is divided into two parts connected in cascade, with the first part H1(z) containing only the zeroes, and the second part H2(z) containing only the poles.

$$H(z) = H_1(z)H_2(z)$$

Where

$$H_1(z) = \sum_{k=0}^{M-1} b_k z^{-k}$$

And

$$H_2(z) = \frac{1}{1 + \sum_{k=1}^{N-1} a_k z^{-k}}$$

These equations can be rewritten as

$$H_1(z) = \frac{W(z)}{X(z)} = \sum_{k=0}^{M} b_k z^{-k}$$

or in time domain

$$w[n] = b_0 x[n] + b_1 x[n-1] + \cdots\cdots + b_M x[n-M]$$

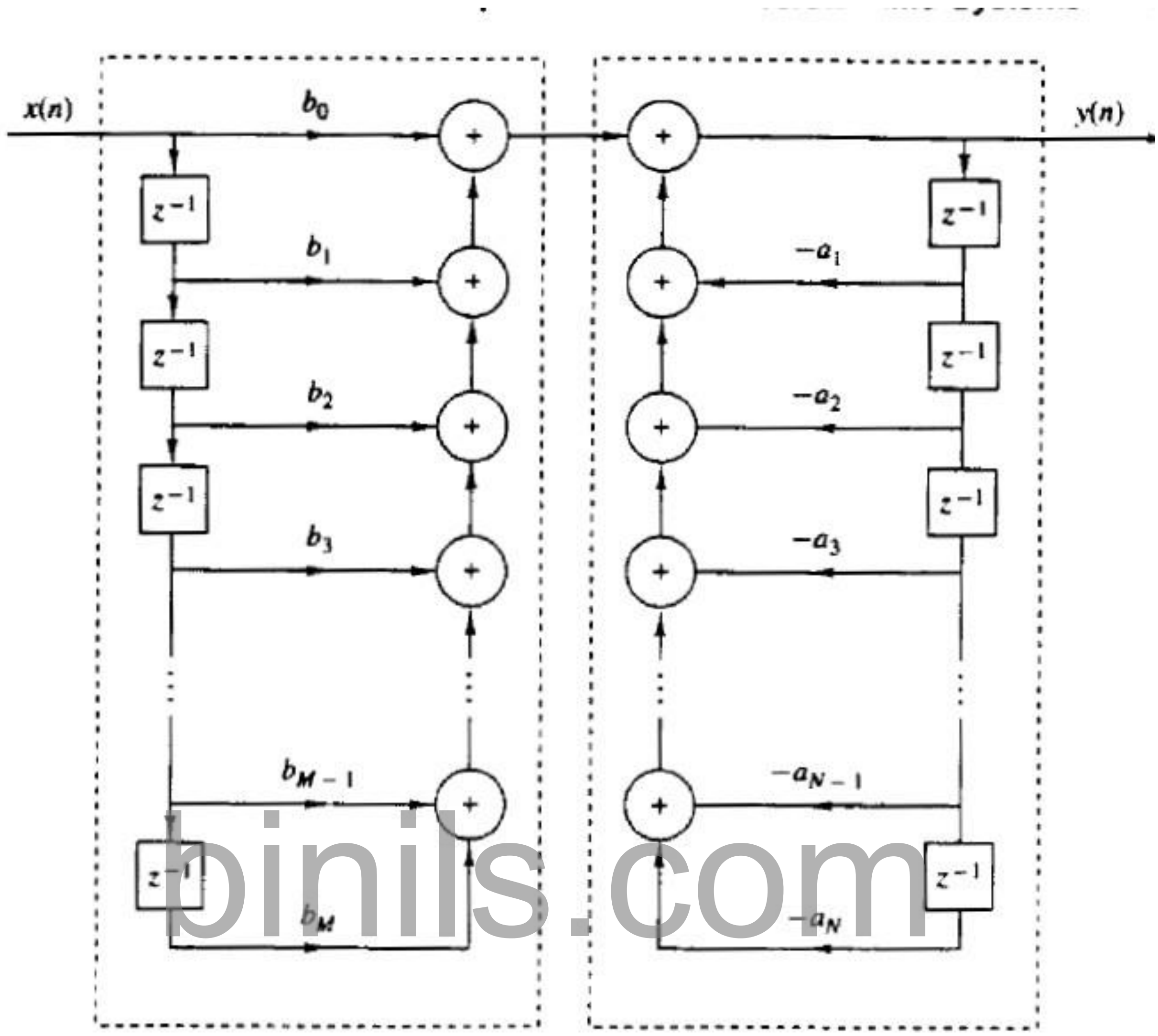
And

$$H_2(z) = \frac{Y(z)}{W(z)} = \frac{1}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

or in time domain

$$y[n] = w[n] - a_1 y[n-1] - a_2 y[n-2] - \cdots\cdots - a_N y[n-N]$$

Realizing the above two equations for H1(z) and H2(z) using basic building blocks and connecting them in cascade, we obtain the **Direct Form – I structure** as follows



DIRECT FORM II REALIZATION

It is the direct implementation of transfer function describing the system. Instead of using separate unit delay elements for input and output variables separately, an intermediate variable is unit delay element. It provides direct relation between time domain and z-domain equations.

Advantages:

- It uses minimum number of unit delay element Straight forward realization

Since, in a cascade arrangement, the order of the systems is not important, the all – pole system H2(z) and the all - zero system H1(z) can be interchanged.i.e.,

$$H_1(z) = \frac{V(z)}{X(z)} = \frac{1}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

Or, in time domain,

$$v[n] = x[n] - a_1 v[n-1] - a_2 v[n-2] - \cdots\cdots - a_N v[n-N]$$

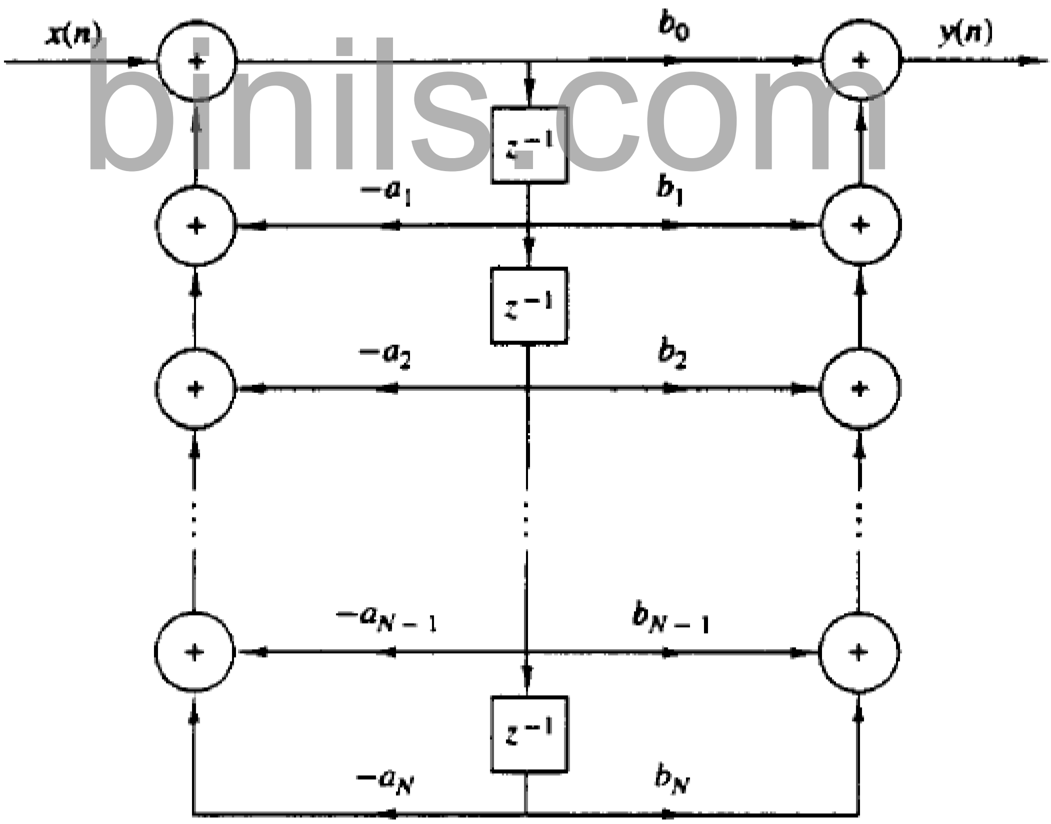
And

$$H_1(z) = \frac{Y(z)}{V(z)} = \sum_{k=0}^{M} b_k z^{-k}$$

Or, in time domain,

$$y[n] = b_0 v[n] + b_1 v[n-1] + \cdots\cdots + b_M v[n-M]$$

Both the time domain equations involve the delayed versions of v[n], and hence require a single set of delay elements. This results in the **Direct – Form II structure** as follows



CASCADE FORM (SERIES FORM)

In cascade form realization the given transfer function is expressed as a product of several transfer function and each of these transfer function is realized in direct form II and then all those realized structures are cascaded i.e., connected in series.

Consider a system with the following system function

$$H(Z) = \frac{(b_{k0} + b_{k1}Z^{-1} + b_{k2}Z^{-2})(b_{m0} + b_{m1}Z^{-1} + b_{m2}Z^{-2})}{(1 + a_{k1}Z^{-1} + a_{k2}Z^{-2})(1 + a_{m1}Z^{-1} + a_{m2}Z^{-2})} = H_1(Z)H_2(Z)$$

Where,

$$H_1(Z) = \frac{(b_{k0} + b_{k1}Z^{-1} + b_{k2}Z^{-2})}{(1 + a_{k1}Z^{-1} + a_{k2}Z^{-2})}$$

$$H_2(Z) = \frac{(b_{m0} + b_{m1}Z^{-1} + b_{m2}Z^{-2})}{(1 + a_{m1}Z^{-1} + a_{m2}Z^{-2})}$$

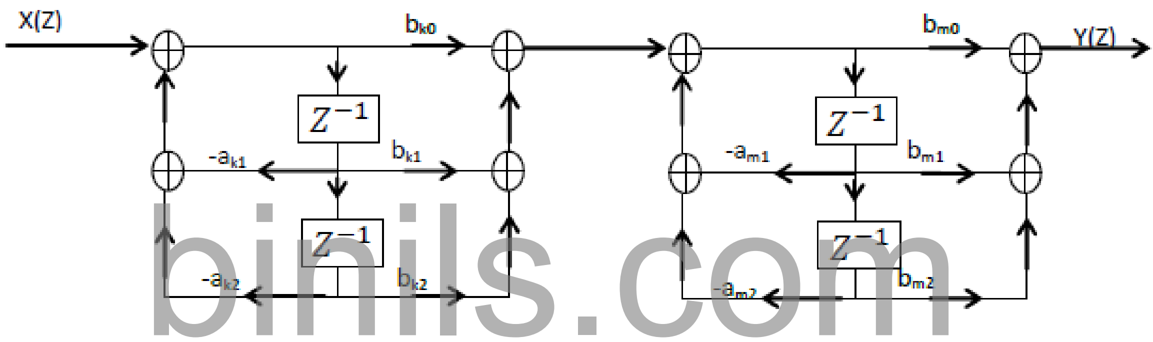


Fig Cascade form realization

PARALLEL FORM REALIZATION

The given transfer function is expressed into its partial fractions and each factor is realized in direct form II and all those realized structures are connected in parallel as shown in Fig

Consider a system with the following system function

$$H(Z) = C + \sum_{k=1}^{N} \frac{C_k}{1 - p_k Z^{-1}}$$

Where $\{P_k\}$ are poles of the system function

$$H(Z) = C + \frac{C_1}{1 - P_1 Z^{-1}} + \frac{C_2}{1 - P_2 Z^{-1}} + \cdots \ldots \ldots \ldots . . + \frac{C_N}{1 - P_N Z^{-1}}$$

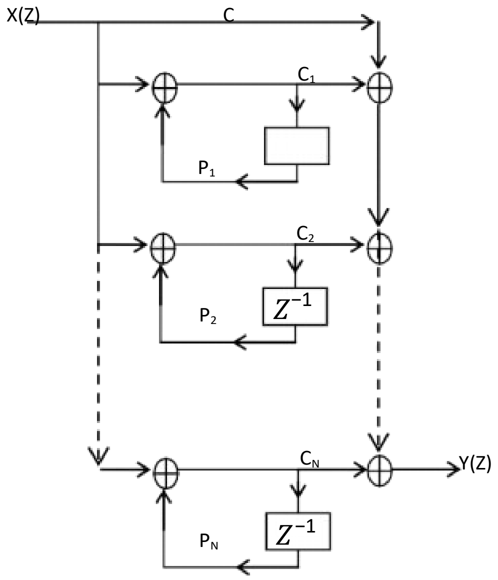


Fig: Parallel form realization

5.3 IMPULSE RESPONSE

When the input to a discrete time system is a unit impulse $\delta(n)$ then the output is called an impulse response of the system and is denoted by $h(n)$

$\therefore$Impulse response $h(n) = H\{\delta(n)\}$

$$\delta(n) \rightarrow \boxed{H} \rightarrow h(n)$$

Impulse response of interconnected systems

Parallel connections of discrete time systems (Distributive property)

Consider two LTI systems with impulse response h1(n) and h2(n) connected in parallel asshown in Fig

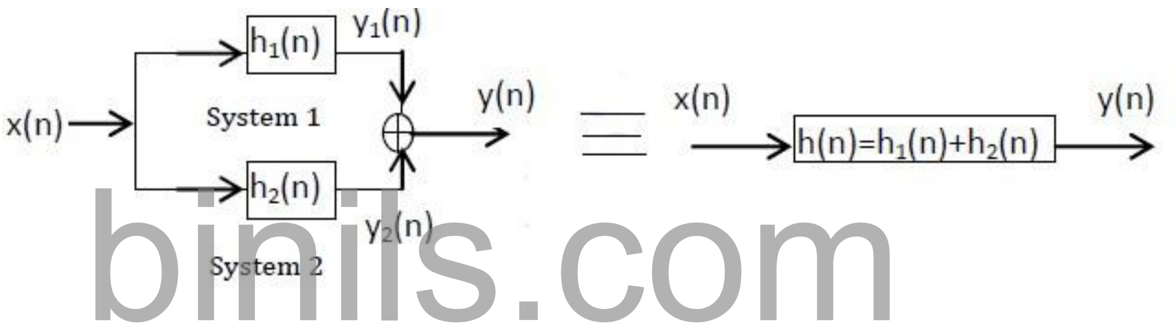
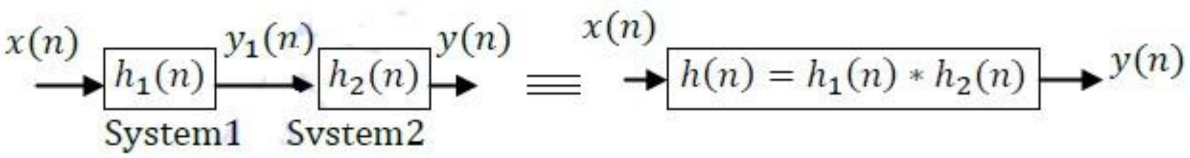


Fig Parallel connections of discrete time systems

Cascade connection of discrete time systems (Associative property)



Let us consider two systems with impulse $h_1(n)$ and $h_2(n)$ connected in cascade as shown in Fig

Example 1 : Determine the frequency response and impulse response

$$y(n) - \frac{1}{6}y(n-1) - \frac{1}{6}y(n-2) = x(n)$$

Solution:

$$y(n) - \frac{1}{6}y(n-1) - \frac{1}{6}y(n-2) = x(n)$$

Applying DTFT

$$Y(e^{j\omega})-\frac{1}{6}e^{-j\omega}Y(e^{j\omega})-\frac{1}{6}e^{-2j\omega}Y(e^{j\omega})=X(e^{j\omega})$$

Frequency Response

$$H(e^{j\omega})=\frac{Y(e^{j\omega})}{X(e^{j\omega})}=\frac{1}{1-\frac{1}{6}e^{-j\omega}-\frac{1}{6}e^{-2j\omega}}=\frac{e^{2j\omega}}{e^{2j\omega}-\frac{1}{6}e^{j\omega}-\frac{1}{6}}$$

$$\frac{H(e^{j\omega})}{e^{j\omega}}=\frac{e^{j\omega}}{e^{2j\omega}-\frac{1}{6}e^{j\omega}-\frac{1}{6}}=\frac{A}{e^{j\omega}-\frac{1}{2}}+\frac{B}{e^{j\omega}+\frac{1}{3}}$$

$$e^{j\omega}=A\left(e^{j\omega}+\frac{1}{3}\right)+B\left(e^{j\omega}-\frac{1}{2}\right)$$

At $e^{j\omega}=-\frac{1}{3}$

$-\frac{1}{3}=B\left(-\frac{1}{3}-\frac{1}{2}\right)$, $\therefore B=\frac{2}{5}$

At $e^{j\omega}=\frac{1}{2}$

$\frac{1}{2}=A\left(\frac{1}{2}+\frac{1}{3}\right)$, $\therefore A=\frac{3}{5}$

$$H(e^{j\omega})=\frac{\frac{3}{5}e^{j\omega}}{e^{j\omega}-\frac{1}{2}}+\frac{\frac{2}{5}e^{j\omega}}{e^{j\omega}+\frac{1}{3}}$$

Applying Inverse DTFT,

$$h(n)=\frac{3}{5}\left(\frac{1}{2}\right)^n u(n)+\frac{2}{5}\left(-\frac{1}{3}\right)^n u(n)$$

Example 2:Find response of system using DTFT.

$$h(n)=\left(\frac{1}{2}\right)^n u(n), x(n)=\left(\frac{3}{4}\right)^n u(n).$$

Solution:

$$h(n)=\left(\frac{1}{2}\right)^n u(n), x(n)=\left(\frac{3}{4}\right)^n u(n).$$

$$H(e^{j\omega})=\frac{1}{1-\frac{1}{2}e^{-j\omega}} \quad ; \quad X(e^{j\omega})=\frac{1}{1-\frac{3}{4}e^{-j\omega}}$$

$$Y(e^{j\omega})=H(e^{j\omega})X(e^{j\omega})$$

$$Y(e^{j\omega}) = \frac{1}{1-\frac{1}{2}e^{-j\omega}} \cdot \frac{1}{1-\frac{3}{4}e^{-j\omega}} = \frac{e^{j\omega}}{e^{j\omega}-\frac{1}{2}} \cdot \frac{e^{j\omega}}{e^{j\omega}-\frac{3}{4}}$$

$$\frac{Y(e^{j\omega})}{e^{j\omega}} = \frac{e^{j\omega}}{\left(e^{j\omega}-\frac{1}{2}\right)\left(e^{j\omega}-\frac{3}{4}\right)} = \frac{A}{e^{j\omega}-\frac{1}{2}} + \frac{B}{e^{j\omega}-\frac{3}{4}}$$

$$e^{j\omega} = A\left(e^{j\omega}-\frac{3}{4}\right) + B\left(e^{j\omega}-\frac{1}{2}\right)$$

At $e^{j\omega} = \frac{1}{2}$

$\frac{1}{2} = A\left(\frac{1}{2}-\frac{3}{4}\right)$, $\therefore A = -2$

At $e^{j\omega} = \frac{3}{4}$

$\frac{3}{4} = B\left(\frac{3}{4}-\frac{1}{2}\right)$, $\therefore B = 3$

$$\frac{Y(e^{j\omega})}{e^{j\omega}} = \frac{-2}{e^{j\omega}-\frac{1}{2}} + \frac{3}{e^{j\omega}-\frac{3}{4}} \quad => \quad Y(e^{j\omega}) = \frac{-2e^{j\omega}}{e^{j\omega}-\frac{1}{2}} + \frac{3e^{j\omega}}{e^{j\omega}-\frac{3}{4}}$$

Applying DTFT,

$$y(n) = -2\left(\frac{1}{2}\right)^n u(n) + 3\left(\frac{3}{4}\right)^n u(n)$$

5.4 CONVOLUTION SUM

The convolution sum provides a concise, mathematical way to express the output of an LTI system based on an arbitrary discrete-time input signal and the system's response. The convolution sum is expressed as

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

1. Convolution is commutative

$$x[n] * h[n] = h[n] * x[n]$$

2. Convolution is Distributive

$$x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n]$$

3. System connected in cascade

$$y[n] = h_1[n] * [\, h_2[n] * x[n] \,] = [\, h_1[n] * h_2[n] \,] * x[n]$$

4. System connected in parallel\

$$y[n] = h_1[n] * x[n] + h_2[n] * x[n] \,] = [\, h_1[n] + h_2[n] \,] * x[n]$$

LTI Systems are said to be stable if ,

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

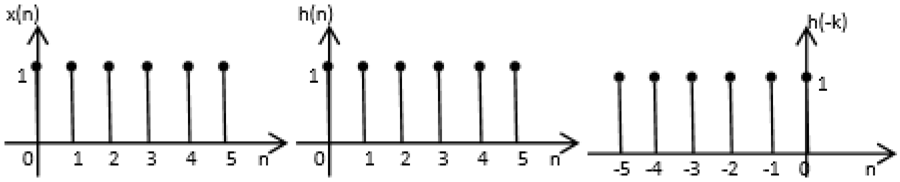
LTI system are causal if,

$$h(n) = 0\,, n < 0.$$

EXAMPLE 1: Convolve the following discrete time signals using graphical convolution $x(n) = h(n) = u(n)$.

Solution:

$$x(n) = u(n) = 1; n \geq 0$$
$$h(n) = u(n) = 1; n \geq 0$$



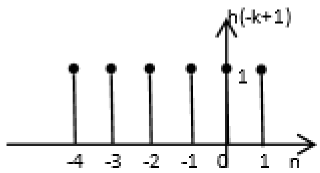
$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

$$when\; n = 0$$

$$y(0) = (1)(1) = 1$$

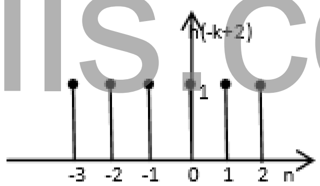
$$when\; n = 1$$

$$y(1) = (1)(1) + (1)(1) = 2$$



$$when\; n = 2$$

$$y(2) = (1)(1) + (1)(1) + (1)(1) = 3$$



$$\therefore y(n) = \{1,2,3,4,5,\ldots\}$$

Example 2: Compute linear convolution $x(n) = \{2,2,0,1,1\}$ $h(n) = \{1,2,3,4\}$.

Solution:

	2	2	0	1	1
1	2	2	0	1	1
2	4	4	0	2	2
3	6	6	0	3	3
4	8	8	0	4	4

$$y(n) = x(n) * h(n) = \{2,6,10,15,11,5,7,4\}$$

5.5 LTI SYSTEM ANALYSIS USING DTFT

Output of LTI system is given by linear convolution

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

Let the system be excited by the sinusoidal or phaser $e^{j\omega n}$.

$$\therefore x(n) = e^{j\omega n}\ for -\infty < n < \infty$$

Hence the signal is complex in nature .It has unit amplitude and frequency is $'\omega'$.The output is given by

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)e^{j\omega(n-k)}$$

$$== \sum_{k=-\infty}^{\infty} h(k)e^{j\omega n}.\, e^{j\omega k}$$

$$= [\sum_{k=-\infty}^{\infty} h(k)e^{j\omega k}]\, e^{j\omega n}$$

$$= H(\omega)e^{j\omega n}$$

$$H(\omega) = \sum_{k=-\infty}^{\infty} h(k)e^{j\omega k}$$

$H(\omega)$ is the Fourier transform of $h(k)$ and $h(k)$ is the unit sample response. $H(\omega)$ is called the transfer function of the system. $H(\omega)$ is complex valued function of ω in the range $-\pi \le \omega \le \pi$.The transfer function of $H(\omega)$ can be expressed in polar form as

$$H(\omega) = |H(\omega)|e^{j\, \llcorner H(\omega)}$$

$$|H(\omega)|\ is\ the\ magnitude\ of\ H(\omega)$$

$$\llcorner H(\omega)\ is\ the\ angle\ of\ H(\omega)$$

LTI SYSTEM ANALYSIS USING Z-TRANSFORM

The Z-Transform of impulse response is called transfer or system function H(Z).

$$Y(Z) = X(Z)H(Z)$$

General form of LCCDE

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

Computing the Z-Transform

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

Example 1: Consider the system described by the difference equation.

$$y[n] = x[n] + \frac{1}{3}x[n-1] + \frac{5}{4}y[n-1] - \frac{1}{2}y[n-2] + \frac{1}{16}y[n-3]$$

Solution:

$$y[n] = x[n] + \frac{1}{3}x[n-1] + \frac{5}{4}y[n-1] - \frac{1}{2}y[n-2] + \frac{1}{16}y[n-3]$$

Here $N=3,\ M=1$. Order 3 homogeneous equation:

$$y[n]-\frac{5}{4}y[n-1]+\frac{1}{2}y[n-2]-\frac{1}{16}y[n-3]=0 \qquad n\geq 2$$

The characteristic equation:

$$1-\frac{5}{4}a^{-1}+\frac{1}{2}a^{-2}-\frac{1}{16}a^{-3}=0$$

The roots of this third order polynomial is: $a_1=a_2=1/2 \quad a_3=1/4$ and

$$y_h[n]=h[n]=A_1(\frac{1}{2})^n+A_2n(\frac{1}{2})^n+A_3(\frac{1}{4})^n, \quad n\geq 2$$

Let us assume $y[-1]=0$ then (3.52) for this case becomes:

$$\begin{bmatrix} a_0 & 0 \\ a_1 & a_0 \end{bmatrix}\cdot\begin{bmatrix} y[0] \\ y[1] \end{bmatrix}=\begin{bmatrix} b_0 \\ b_1 \end{bmatrix} \;\Rightarrow\; \begin{bmatrix} 1 & 0 \\ -5/4 & 1 \end{bmatrix}\cdot\begin{bmatrix} y[0] \\ y[1] \end{bmatrix}=\begin{bmatrix} 1 \\ 1/3 \end{bmatrix} \;\Rightarrow y[0]=1;\ y[1]=19/12$$

with these we have the impulse response of this system:

$$h[n]=-\frac{4}{3}(\frac{1}{2})^n+\frac{10}{3}n(\frac{1}{2})^n+\frac{7}{3}(\frac{1}{4})^n, \quad n\geq 0$$