

5.1 CONCEPT OF STATE VARIABLES

State space analysis is an excellent method for the design and analysis of control systems. The conventional and old method for the design and analysis of control systems is the transfer function method. The transfer function method for design and analysis had many drawbacks.

Drawbacks of transfer function model analysis:

- a. Transfer function is defined under zero initial conditions
- b. Transfer function is applicable to linear time invariant systems
- c. Transfer function analysis is restricted to single input and single output systems
- d. Does not provide information regarding the internal state of the system

Advantages of state variable analysis:

- It can be applied to linear system
- It can be applied to non-linear system
- It can be applied to time varying system
- It can be applied to time invariant system
- It can be applied to multiple input multiple output system
- Its gives idea about the internal state of the system

A state variable is one of the set of variables that are used to describe the mathematical "state" of a dynamical system. Intuitively, the state of a system describes enough about the system to determine its future behaviour in the absence of any external forces affecting the system. The state variable analysis can be applied for any type of systems. In this method of analysis, it is not necessary that the state variables represent physical quantities of the system, but variables that do not represent physical quantities and those that are neither measurable nor observable may be chosen as state variables.

STATE SPACE FORMULATION

State:

The state of a dynamic system is the minimal set of variables called state variables such that the knowledge of these variables at time $t = t_0$ (initial condition), together with the knowledge of input for $t \geq t_0$, completely determines the behaviour of the system for any time $t > t_0$. (or) A set of variables which describes the system at any time instant are called state variables. In the state variable formulation of a system, in general, a system consists of m -inputs, p -outputs and n -state variables. The state space representation of the system may be visualized as shown in figure 5.1.1.

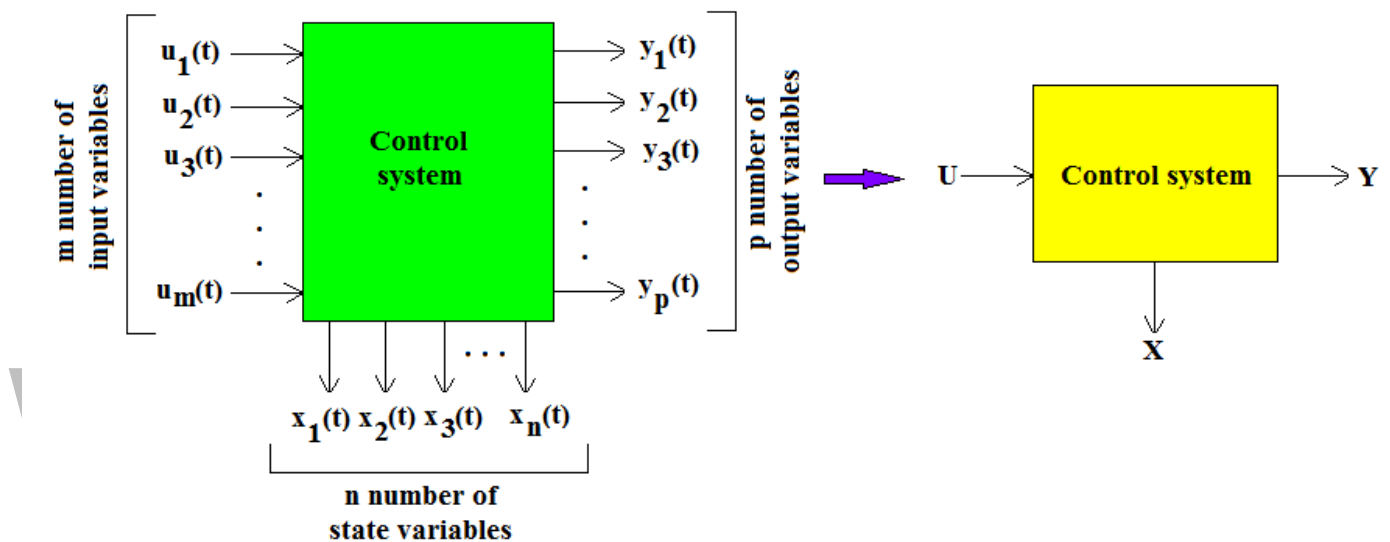


Figure 5.1.1 State space representation of a system

[Source: "Control Systems" by A. Nagoor Kani, Page: 5.2]

Let us consider a multi input & multi output (MIMO) system is having

m inputs: $u_1(t), u_2(t), \dots, u_m(t)$

p number of outputs: $y_1(t), y_2(t), \dots, y_p(t)$

n number of state variables: $x_1(t), x_2(t), \dots, x_n(t)$

The different variables may be represented by the vectors (column matrix) as shown below:

Input vector

$$U(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_m(t) \end{bmatrix}$$

Output vector

$$Y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_p(t) \end{bmatrix}$$

State variable vector

$$X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

State vector:

If n state variables are needed to completely describe the behaviour of a given system, then these n state variables can be considered the n components of a vector X. Such a vector is called a state vector.

State space:

The n-dimensional space whose co-ordinate axes consists of the x_1 axis, x_2 axis.....
 x_n axis, where $x_1, x_2, , x_n$ are state variables is called a state space.

5.4 CONCEPTS OF CONTROLLABILITY AND OBSERVABILITY

CONCEPT OF CONTROLLABILITY

A system is said to be completely controllable, if it is possible to transfer the system state from any initial state $x(t_0)$ to any desired state $x(t)$ in specified finite time by a control vector $u(t)$.

If any of the state variable is independent of the control $u(t)$, there would be no way of driving this particular state variable to desired state in finite time by means of control effort. Therefore, this particular state is said to be uncontrollable. As long as there is at least one uncontrollable state, the system is said to be not completely controllable or 'uncontrollable'.

Consider a single input, linear time invariant system:

$$\dot{X}(t) = AX(t) + BU(t)$$

Let the initial system state be $x(0)$ and the final state be $x(t_f)$. The system is controllable if it is possible to construct a control signal, which in finite time interval $0 < t \leq t_f$, will transfer the system state from $x(0)$ to $x(t_f)$. The above equation is completely controllable if and only if the rank of the composite matrix is n .

$$Q_C = [B \ : \ AB \ : \ \dots \ : \ A^{n-1}B]$$

Since only matrices A and B are involved, we may say that the pair $(A;B)$ is controllable if rank of Q_C is n .

CONCEPT OF OBSERVABILITY

A system is said to be completely observable, if every state $x(t_0)$ can be completely identified by measurement of outputs $y(t)$ over a finite time interval. Given a LTI system that is described by the dynamic equations, the state $x(t_0)$ is said to be observable if given any input $u(t)$, there exists a finite time $t_f \geq t_0$ such that knowledge of $u(t)$ for $t_0 \leq t < t_f$, matrices A, B, C , & D and the output $y(t)$; for $t_0 \leq t < t_f$ are sufficient to determine $x(t_0)$. The necessary and sufficient condition for the system to be completely observable is necessary and sufficient that the following $n \times n_p$ observability matrix has rank of n .

$$Q_o = [C^T \quad A^T C^T \quad (A^2)^T C^T \quad \dots \quad (A^{n-1})^T C^T]$$

5.3 SOLUTION OF STATE AND OUTPUT EQUATION IN CONTROLLABLE CANONICAL FORM

Consider the state equation of a linear time invariant system as,

$$\dot{X}(t) = AX(t) + BU(t)$$

The matrices A and B are constant matrices. This state equation can be of two types,

1. Homogeneous
2. Non-homogeneous

HOMOGENEOUS EQUATION

If A is a constant matrix and input control forces are zero then the equation takes the form

$$\dot{X}(t) = AX(t)$$

Such an equation is called homogeneous equation. The obvious equation is considered if input is zero. In such systems, the driving force is provided by the initial conditions of the system to produce the output. For example, consider a series RC circuit in which a capacitor is initially charged to V volts. The current is the output. Now there is no input control force, i.e., external voltage applied to the system. But the initial voltage on the capacitor drives the current through the system and capacitor starts discharging through the resistance, R. such a system works on the initial conditions without any input applied to it is called homogeneous system.

NON-HOMOGENEOUS EQUATION

If A is a constant matrix and matrix U(t) is non-zero vector i.e. the input control forces are applied to the system then the equation takes normal form as,

$$\dot{X}(t) = AX(t) + BU(t)$$

Such an equation is called non-homogeneous equation. Most of the practical systems require inputs to drive them. Such systems are nonhomogeneous linear systems. The solution of the state equation is obtained by considering basic method of finding the solution of homogeneous equation.

STATE TRANSITION MATRIX

Properties of State Transition Matrix

1. $\phi(0) = e^{A \times 0} = I$ (unit matrix)
2. $\phi(t) = e^{At} = (e^{-At})^{-1} = [\phi(-t)]^{-1}$
or $\phi^{-1}(t) = \phi(-t)$
3. $\phi(t_1 + t_2) = e^{A(t_1+t_2)} = e^{At_1} e^{At_2} = \phi(t_1)\phi(t_2)$

Computation of State transition matrix

The state transition matrix, e^{At} can be computed by any one of the following two methods:

Method 1: Computation of e^{At} using matrix exponential

If the system matrix 'A' is an (n×n) square matrix, then each of these exponentials is an (n×n) square matrix of time functions, and one of the consequences of a theorem developed in linear algebra, known as the Cayley-Hamilton theorem, shows that such a matrix may be expressed as an (n-1)st degree polynomial in the matrix A.

That is,

$$e^{At} = I + At + \frac{1}{2!} A^2 t^2 + \frac{1}{3!} A^3 t^3 + \dots + \frac{1}{i!} A^i t^i$$

where, e^{At} – State transition matrix of order n x n

A – System matrix of order n x n

I – Unit matrix of order n x n

Method 2: Computation of e^{At} using Laplace transform

The theorem also states that the equation remains an equality if I is replaced by unity and A is replaced by any one of the scalar roots sI of the nth-degree scalar equation, $\det(sI - A) = 0$. The expression $\det(sI - A)$ indicates the determinant of the matrix (sI-A). This determinant is an nth-degree polynomial in s. Let us assume that then roots are all different. This equation is called the characteristic equation of the matrix a, and the values of s which are the roots of the equation are known as the eigen values of A.

Consider the state equation without input vector,

$$\dot{X}(t) = AX(t)$$

On taking Laplace transform, we get,

$$sX(s) - X(0) = AX(s)$$

$$sX(s) - AX(s) = X(0)$$

$$\begin{aligned} sIX(s) - AX(s) &= X(0) \quad (sI \\ &- A)X(s) = X(0) \end{aligned}$$

Pre-multiplying both sides by $(sI - A)^{-1}$,

$$X(s) = (sI - A)^{-1}X(0)$$

On taking inverse Laplace transform,

$$x(t) = L^{-1}[(sI - A)^{-1}]x(0)$$

On comparing with solution of state equation,

$$e^{At} = L^{-1}[(sI - A)^{-1}]$$

Also,

$$e^{At} = \phi(t)$$

where,

$$\phi(s) = (sI - A)^{-1}$$

which is the resolvent matrix.

Consider the state equation with input vector,

$$\dot{X}(t) = AX(t) + BU(t)$$

On taking Laplace transform, we get,

$$sX(s) - X(0) = AX(s) + BU(s)$$

$$\begin{aligned} sIX(s) - AX(s) &= X(0) + BU(s) \quad (sI \\ &- A)X(s) = X(0) + BU(s) \end{aligned}$$

Pre-multiplying both sides by $(sI - A)^{-1}$,

$$X(s) = (sI - A)^{-1}X(0) + (sI - A)^{-1}BU(s)$$

$$X(s) = \phi(s)X(0) + \phi(s)BU(s)$$

On taking inverse Laplace transform,

$$x(t) = \phi(t)x(0) + L^{-1}[\phi(s)BU(s)]$$

Solution of output equation by Laplace Transform

$$Y(s) = CX(s) + DU(s)$$

$$y(t) = L^{-1}[CX(s) + DU(s)]$$

CONTROLLABLE CANONICAL FORM (CCF)

Probably the most straightforward method for converting from the transfer function of a system to a state space model is to generate a model in "controllable canonical form."

Consider a system defined by,

$$y^{(n)} + a_1y^{(n-1)} + \dots + a_{n-1}y' + a_ny = b_0u^{(n)} + b_1u^{(n-1)} + \dots + b_{n-1}u' + b_nu$$

where u is the control input and y is the output. It can be written as,

$$\frac{Y(s)}{U(s)} = \frac{[b_0s^n + b_1s^{n-1} + \dots + b_{n-1}s + b_n]}{[s^n + a_1s^{n-1} + a_{n-1}s + a_n]}$$

Controllable canonical form of this system is given by,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \\ -a_n & -a_{n-1} & -a_{n-2} & \dots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$

$$y = [b_n - a_nb_0 \quad b_{n-1} - a_{n-1}b_0 \quad \dots \quad \dots \quad b_2 - a_2b_0 \quad b_1 - a_1b_0] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + b_0u$$

OBSERVABLE CANONICAL FORM

The observable canonical form of the state-space representation of this system is given by

DIAGONAL CANONICAL FORM

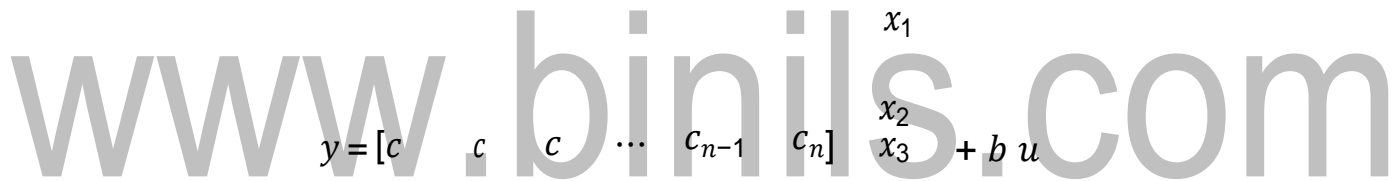
There are cases where the dominator polynomial involves only distinct roots. For the distinct root case, we can write the equation in the form of

$$\frac{Y(s)}{U(s)} = \frac{[b_0s^n + b_1s^{n-1} + \dots + b_{n-1}s + b_n]}{(s+p_1)(s+p_2)\dots(s+p_n)} = b_0 + \frac{c_1}{s+p_1} + \frac{c_2}{s+p_2} + \dots + \frac{c_n}{s+p_n}$$

The diagonal canonical form of the state-space representation of this system is given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -p_1 & 0 & 0 & \dots & 0 \\ 0 & -p_2 & 0 & \dots & 0 \\ 0 & 0 & -p_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & -p_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \\ -p_n \end{bmatrix} u$$

$$y = [c_1 \ c_2 \ c_3 \ \dots \ c_{n-1} \ c_n] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + b u$$



5.2 STATE MODELS FOR LINEAR AND TIME INVARIANT SYSTEMS

State model is given by state and output equation

State equation:

$$\dot{X}(t) = AX(t) + BU(t)$$

Output equation:

$$Y(t) = CX(t) + DU(t)$$

where,

A is state matrix of size (n x n)

B is the input matrix of size (n x m)

C is the output matrix of size (p x n)

D is the direct transmission matrix of size (p x m)

X(t) is the state vector of size (n x 1)

Y(t) is the output vector of size (p x 1)

U(t) is the input vector of size (m x 1)

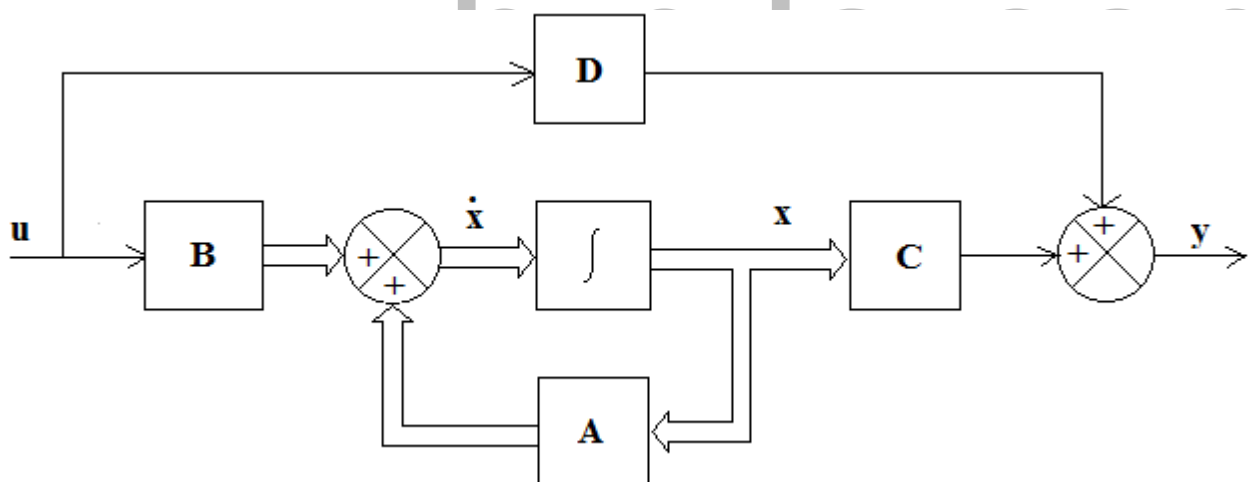


Figure 5.2.1 State space model diagram

[Source: "Modern Control Engineering" by Katsuhiko Ogata, Page: 828]

STATE SPACE REPRESENTATION USING PHYSICAL VARIABLES

In state-space modelling of systems, the choice of state variables is arbitrary. One of the possible choices of state variables. The physical variables of electrical systems are current or voltage in the R, L and C elements. The physical variables of mechanical systems are displacement, velocity and acceleration. The advantages of choosing the physical variables (or quantities) of the system as state variables are the following,

- 1. The state variables can be utilized for the purpose of feedback*
- 2. The implementation of design with state variable feedback becomes straight forward*
- 3. The solution of state equation gives time variation of variables which have direct relevance to the physical system.*

The drawback in choosing the physical quantities as state variables is that the solution of state equation may be a difficult task. In state space modelling using physical variables, the state equations are obtained from a basic model of the system which is developed using the fundamental elements of the system.

Electrical System

The basic model of an electrical system can be obtained by using the fundamental elements Resistor, Capacitor and Inductor. Using these elements, the electrical network or equivalent circuit of the system is drawn. Then the differential equations governing the electrical systems can be formed by writing Kirchhoff's Current Law equations by choosing various nodes in the network or Kirchhoff's Voltage Law by choosing various closed path in the network. A minimal number of state variables are chosen for obtaining the state model of the system. The best choice of state variables in electrical system are currents and voltages in energy storage elements. The energy storage elements are inductance and capacitance. The physical variables in the differential equations are replaced by state variables and the equations are rearranged as first order differential equations. These set of first order equations constitutes the state equation of the system. The inputs to the system are voltage sources or current sources. The outputs in electrical system are usually voltages or currents in energy dissipating elements. The resistance is energy dissipating element in electrical network. In general, the output variables can be any voltage or current in the network.

Mechanical Translational System

The basic model of mechanical translational system can be obtained by using three basic elements; mass, spring and dash-pot. When a force is applied to a mechanical translational system, it is opposed by opposing forces due to mass, friction and elasticity of the system. The forces acting on a body are governed by Newton's second law of motion. The differential equations governing the system are obtained by writing force balance equations at various nodes in the system. A node is a meeting point of elements.

Guidelines to form the state model of mechanical translational systems

1. For each node in the system one differential equation can be framed by equating the sum of applied forces to the sum of opposing forces. Generally, the nodes are mass elements of the system, but in some cases the nodes may be without mass element.
2. Assign a displacement to each node and draw a free body diagram for each node. The free body diagram is obtained by drawing each mass of node separately and then marking all the forces acting on it.
3. In the free body diagram, the opposing forces due to mass, spring and dash –pot are always act in a direction opposite to applied force. The displacement, velocity and acceleration will be in the direction of applied force or in the direction opposite to that of opposing force.
4. For each free body diagram write one differential equation by equating the sum of applied forces to the sum of opposing forces.
5. Choose a minimum number of state variables. The choice of state variables are displacement, velocity or acceleration.
6. The physical variables in differential equations are replaced by state variables and the equations are rearranged as first order differential equations. These set of first order equations constitute the state equation of the system.
7. The inputs are the applied forces and the outputs are the displacement, velocity or acceleration of the desired nodes.

Mechanical Rotational System

The basic model of mechanical rotational system can be obtained by using three basic elements moment of inertia of mass, rotational dash-pot and rotational spring. When a torque is applied to a mechanical rotational system, it is opposed by opposing torques due to moment of inertia, friction and elasticity of the system. The torque acting on a body are governed by Newton's second law of motion. The differential equations governing the system are obtained by writing torque balance equations at various nodes in the system. A node is a meeting point of elements.

Guidelines to form the state model of mechanical rotational systems

1. For each node in the system one differential equation can be framed by equating the sum of applied torques to the sum of opposing torques. Generally, the nodes are mass elements of the system, but in some cases the nodes may be without mass element.
2. Assign an angular displacement to each node and draw a free body diagram for each node. The free body diagram is obtained by drawing each mass of node separately and them marking all the torques acting on it.
3. In the free body diagram, the opposing torques due to mass of inertia, spring and dashpot are always act in a direction opposite to applied force. The angular displacement, velocity and acceleration will be in the direction of applied torque or in the direction opposite to that of opposing torque.
4. For each free body diagram write one differential equation by equating the sum of applied torques to the sum of opposing torques.
5. Choose a minimum number of state variables. The choice of state variables are angular displacement, velocity or acceleration.
6. The physical variables in differential equations are replaced by state variables and the equations are rearranged as first order differential equations. These set of first order equations constitute the state equation of the system.
7. The inputs are the applied torques and the outputs are the angular displacement, velocity or acceleration of the desired nodes.

STATE SPACE REPRESENTATION USING PHASE VARIABLES

The phase variables are defined as those particular state variables which are obtained from one of the system variables and its derivatives. There are three methods of modelling a system using phase variables. They are,

METHOD 1

Consider the following n^{th} order linear differential equation relating the output $y(t)$ to the input $u(t)$ of a system,

$$y^n + a_1y^{n-1} + a_2y^{n-2} + \dots + a_{n-2}\ddot{y} + a_{n-1}\dot{y} + a_ny = bu$$

By choosing the output, y and their derivatives as state variables, we get,

$$x_1 = y$$

$$x_2 = \dot{y}$$

$$x_3 = \ddot{y}$$

⋮

$$x_n = y^{n-1}$$

$$\dot{x}_n = y^n$$

$$\dot{x}_n + a_1x_n + a_2x_{n-1} + \dots + a_{n-2}x_3 + a_{n-1}x_2 + a_nx_1 = bu$$

$$\dot{x}_n = -a_1x_n - a_2x_{n-1} - \dots - a_{n-2}x_3 - a_{n-1}x_2 - a_nx_1 + bu$$

The state equations of the system are

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = x_4$$

⋮

$$\dot{x}_{n-1} = x_n$$

$$\dot{x}_n = -a_1x_n - a_2x_{n-1} - \dots - a_{n-2}x_3 - a_{n-1}x_2 - a_nx_1 + bu$$

On arranging the above equations in the matrix form, we get,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & \vdots & 0 \\ 0 & 0 & 1 & 0 & \vdots & 0 \\ 0 & 0 & 0 & 1 & \vdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \vdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & -a_{n-3} & \vdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ b \end{bmatrix} u$$

$$\dot{X} = AX + BU$$

This form of matrix A is known as **Bush form (or) Companion form**.

$$y = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}$$

$$Y = CX$$

METHOD 2

Consider the following nth order linear differential equation relating the output y(t) to the input u(t) of a system,

$$y^n + a_1y^{n-1} + a_2y^{n-2} + \dots + a_{n-2}\ddot{y} + a_{n-1}\dot{y} + a_ny = bu$$

Let n = m = 3

$$\ddot{y} + a_1\dot{y} + a_2y + a_3y = b_0\ddot{u} + b_1\dot{u} + b_2u + b_3u$$

On taking Laplace transform with zero initial conditions, we get,

$$\begin{aligned} s^3Y(s) + a_1s^2Y(s) + a_2sY(s) + a_3Y(s) &= b_0s^3U(s) + b_1s^2U(s) + b_2sU(s) + b_3U(s) \\ + a_2s + a_3]Y(s) &= [b_0s^3 + b_1s^2 + b_2s + b_3]U(s) \end{aligned}$$

$$\frac{Y(s)}{U(s)} = \frac{[b_0s^3 + b_1s^2 + b_2s + b_3]}{[s^3 + a_1s^2 + a_2s + a_3]} = \frac{0 \quad b_0 \quad b_1 \quad b_2 \quad b_3}{s^3 \quad [1 + \frac{a_1}{s} + \frac{a_2}{s^2} + \frac{a_3}{s^3}]} = \frac{0 \quad b_0 \quad b_1 \quad b_2 \quad b_3}{1 - [-\frac{a_1}{s} - \frac{a_2}{s^2} - \frac{a_3}{s^3}]}$$

From Mason’s gain formula, the transfer function of the system is given by,

$$T(s) = \frac{1}{\Delta} \sum_K P_K \Delta_K$$

where, P_K – path gain of Kth forward path

Δ = 1 – (sum of loop gain of all individual loops) + (sum of gain products of all possible combinations of two non-touching loops) –

Δ_K = Δ for that part of the graph which is not touching Kth forward path

The transfer function of the system with four forward paths and three feedback loops (touching each other) is given by,

$$T(s) = \frac{P_1 + P_2 + P_3 + P_4}{1 - (P_{11} + P_{12} + P_{13})}$$

By comparing the above equations,

$$P_1 = b_0; P_2 = \frac{b_1}{s}; P_3 = \frac{b_2}{s^2}; P_4 = \frac{b_3}{s^3}; P_{11} = -\frac{a_1}{s}; P_{12} = -\frac{a_2}{s^2}; P_{13} = -\frac{a_3}{s^3}$$

On arranging the above equations in the matrix form, we get,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -a_1 & 1 & 0 & 0 & \vdots & 0 \\ -a_2 & 0 & 1 & 0 & \vdots & 0 \\ -a_3 & 0 & 0 & 1 & \vdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -a_{n-1} & 0 & 0 & 0 & \vdots & 1 \\ -a_n & 0 & 0 & 0 & \vdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} b_1 - a_1 b_0 \\ b_2 - a_2 b_0 \\ b_3 - a_3 b_0 \\ \vdots \\ b_{n-1} - a_{n-1} b_0 \\ b_n - a_n b_0 \end{bmatrix} u$$

$$\dot{X} = AX + BU$$

$$y = [1 \quad 0 \quad 0 \quad \dots \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + b u$$

$$Y = CX + DU$$

METHOD 3

Consider the following n^{th} order linear differential equation relating the output $y(t)$ to the input $u(t)$ of a system,

$$y^n + a_1 y^{n-1} + a_2 y^{n-2} + \dots + a_{n-2} \ddot{y} + a_{n-1} \dot{y} + a_n y = bu$$

Let $n = m = 3$

$$\ddot{y} + a_1 \dot{y} + a_2 y + a_3 y = b_0 \ddot{u} + b_1 \dot{u} + b_2 u + b_3 u$$

On taking Laplace transform with zero initial conditions, we get,

$$\begin{aligned} s^3 Y(s) + a_1 s^2 Y(s) + a_2 s Y(s) + a_3 Y(s) \\ = b_0 s^3 U(s) + b_1 s^2 U(s) + b_2 s U(s) + b_3 U(s) [s^3 + a_1 s^2 \\ + a_2 s + a_3] Y(s) = [b_0 s^3 + b_1 s^2 + b_2 s + b_3] U(s) \\ \frac{Y(s)}{U(s)} = \frac{[b_0 s^3 + b_1 s^2 + b_2 s + b_3] [s^3 + a_1 s^2 + a_2 s + a_3]}{a_1 s^2 + a_2 s + a_3} \end{aligned}$$

Let,

$$\frac{Y(s)}{U(s)} = \frac{Y(s)}{X_1(s)} \cdot \frac{X_1(s)}{U(s)}$$

$$\frac{X_1(s)}{U(s)} = \frac{1}{[s^3 + a_1 s^2 + a_2 s + a_3]}$$

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$$\frac{Y(s)}{X_1(s)} = [b_0 \quad s^3 + b_1 \quad s^2 + b_2 \quad s + b_3]$$

State Equation

On cross multiplying the equation, we get,

$$\begin{aligned} X_1(s)[s^3 + a_1s^2 + a_2s + a_3] &= U(s) \\ s^3X_1(s) + a_1s^2X_1(s) + a_2sX_1(s) + a_3X_1(s) &= U(s) \\ \ddot{x}_1 + a_1\dot{x}_1 + a_2x_1 + a_3x_1 &= u \end{aligned}$$

Let the state variable be x_1, x_2, x_3 .

$$\begin{aligned} x_2 &= \dot{x}_1 \\ x_3 &= \dot{x}_2 = \ddot{x}_1 \\ \dot{x}_3 &= \ddot{x}_2 \end{aligned}$$

On substituting the state variables, we get,

$$\dot{x}_3 + a_1x_3 + a_2x_2 + a_3x_1 = u$$

The state equations are

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= -a_1x_3 - a_2x_2 - a_3x_1 + u \end{aligned}$$

Output Equation

On cross multiplying the equation, we get,

$$Y(s) = [b_0s^3X_1(s) + b_1s^2X_1(s) + b_2sX_1(s) + b_3X_1(s)]$$

Taking inverse Laplace transform, we get,

$$y = b_0\ddot{x}_1 + b_1\dot{x}_1 + b_2x_1 + b_3x_1$$

On substituting the state variables, we get,

$$y = b_0\dot{x}_3 + b_1x_3 + b_2x_2 + b_3x_1$$

Substituting $\dot{x}_3 = -a_1x_3 - a_2x_2 - a_3x_1 + u$, we get,

$$\begin{aligned} y &= b_0(-a_1x_3 - a_2x_2 - a_3x_1 + u) + b_1x_3 + b_2x_2 + b_3x_1 \\ y &= (b_3 - a_3b_0)x_1 + (b_2 - a_2b_0)x_2 + (b_1 - a_1b_0)x_3 + b_0u \end{aligned}$$

Framing the state and output equation in matrix form, we get,

$$\begin{array}{l}
 x_1 \\
 x_2 \\
 x_3 \\
 \vdots \\
 x_{n-1} \\
 \dot{x}_n
 \end{array}
 =
 \begin{bmatrix}
 0 & 1 & 0 & 0 & \vdots & 0 \\
 0 & 0 & 1 & 0 & \vdots & 0 \\
 0 & 0 & 0 & 1 & \vdots & 0 \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 0 & 0 & 0 & 0 & \vdots & 1 \\
 -a_n & -a_{n-1} & -a_{n-2} & -a_{n-3} & \vdots & -a_1
 \end{bmatrix}
 \begin{bmatrix}
 x_1 \\
 x_2 \\
 x_3 \\
 \vdots \\
 x_{n-1} \\
 x_n
 \end{bmatrix}
 +
 \begin{bmatrix}
 0 \\
 0 \\
 0 \\
 \vdots \\
 0 \\
 1
 \end{bmatrix}
 u
 \quad [1]$$

$$\dot{X} = AX + BU$$

This form of matrix A is known as **Bush form (or) Companion form**.

$$y = [b_n - a_n b_0 \quad b_{n-1} - a_{n-1} b_0 \quad \dots \quad \dots \quad b_2 - a_2 b_0 \quad b_1 - a_1 b_0]
 \begin{bmatrix}
 x_1 \\
 x_2 \\
 \vdots \\
 x_{n-1} \\
 x_n
 \end{bmatrix}
 + b_0 u$$

$$Y = CX$$

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STATE SPACE REPRESENTATION USING CANONICAL VARIABLES

In canonical form (or normal form) of state model, the system matrix A will be a diagonal matrix. The elements on the diagonal are the poles of the transfer function of the system. By partial fraction expansion, the transfer function $Y(s)/U(s)$ of the n^{th} order system can be expressed as,

$$\frac{Y(s)}{U(s)} = b_0 + \frac{C_1}{s + \lambda_1} + \frac{C_2}{s + \lambda_2} + \dots + \frac{C_n}{s + \lambda_n}$$

where, $C_1, C_2, C_3, \dots, C_n$ are residues and $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n$ are roots of denominator polynomial (or poles of the system).

$$\frac{Y(s)}{U(s)} = b_0 + \frac{C_1/s}{1 + \lambda_1/s} + \frac{C_2/s}{1 + \lambda_2/s} + \dots + \frac{C_n/s}{1 + \lambda_n/s}$$

$$Y(s) = b_0 U(s) + \frac{C_1/s}{1 + \lambda_1/s} U(s) + \frac{C_2/s}{1 + \lambda_2/s} U(s) + \dots + \frac{C_n/s}{1 + \lambda_n/s} U(s)$$

The state equation can be framed as,

$$\begin{aligned} \dot{x}_1 &= -\lambda_1 x_1 + u \\ \dot{x}_2 &= -\lambda_2 x_2 + u \\ &\vdots \end{aligned}$$

$$\dot{x}_n = -\lambda_n x_n + u$$

The output equation can be framed as,

$$y = C_1 x_1 + C_2 x_2 + \dots + C_n x_n + b_0 u$$

The canonical form of state model in the matrix form is given by,

$$\begin{aligned} \dot{x}_1 &= -\lambda_1 x_1 + 0 x_2 + 0 x_3 + \dots + 0 x_{n-1} + 0 x_n + 1 u \\ \dot{x}_2 &= 0 x_1 - \lambda_2 x_2 + 0 x_3 + \dots + 0 x_{n-1} + 0 x_n + 1 u \\ \dot{x}_3 &= 0 x_1 + 0 x_2 - \lambda_3 x_3 + \dots + 0 x_{n-1} + 0 x_n + 1 u \\ &\vdots \\ \dot{x}_{n-1} &= 0 x_1 + 0 x_2 + 0 x_3 + \dots - \lambda_{n-1} x_{n-1} + 0 x_n + 1 u \\ \dot{x}_n &= 0 x_1 + 0 x_2 + 0 x_3 + \dots - \lambda_n x_n + 1 u \end{aligned}$$

$$y = [C_1 \quad C_2 \quad C_3 \quad \dots \quad C_{n-1} \quad C_n] \begin{bmatrix} x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + b u$$

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JORDAN CANONICAL FORM

$$A = J = \begin{bmatrix} -\lambda_1 & 0 & 0 & 0 & \vdots & 0 \\ 0 & -\lambda_1 & 0 & 0 & \vdots & 0 \\ 0 & 0 & -\lambda_1 & 0 & \vdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \vdots & 0 \\ [0 & 0 & 0 & 0 & \vdots & -\lambda_n] \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 1 \\ [1] \end{bmatrix}$$

$$\dot{Z} = JZ + \tilde{B}U$$

$$Y = \tilde{C}Z + DU$$

where,

$$J = M^{-1}AM; \quad \tilde{B} = M^{-1}B; \quad \tilde{C} = CM$$

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