

### 3.2 BODE PLOT

The Bode plot is a frequency response plot of the sinusoidal transfer function of a system. One is a plot of the magnitude of a sinusoidal transfer function versus  $\log \omega$ . The other is a plot of the phase angle of a sinusoidal transfer function versus  $\log \omega$ . The main advantage of bode plot is that multiplication of magnitudes can be converted into addition. Also, a simple method for sketching an approximate log-magnitude curve is available. A Bode plot is a (semilog) plot of the transfer function magnitude and phase angle as a function of frequency.

The gain magnitude is many times expressed in terms of decibels (dB) =  $20 \log_{10}A$ .

#### Semilog sheet

Two sets of axes: gain on top, phase below (identical)

Logarithmic frequency axes

Gain axis is logarithmic – either explicitly or as units of decibels(dB)

Phase axis is linear with units of degrees

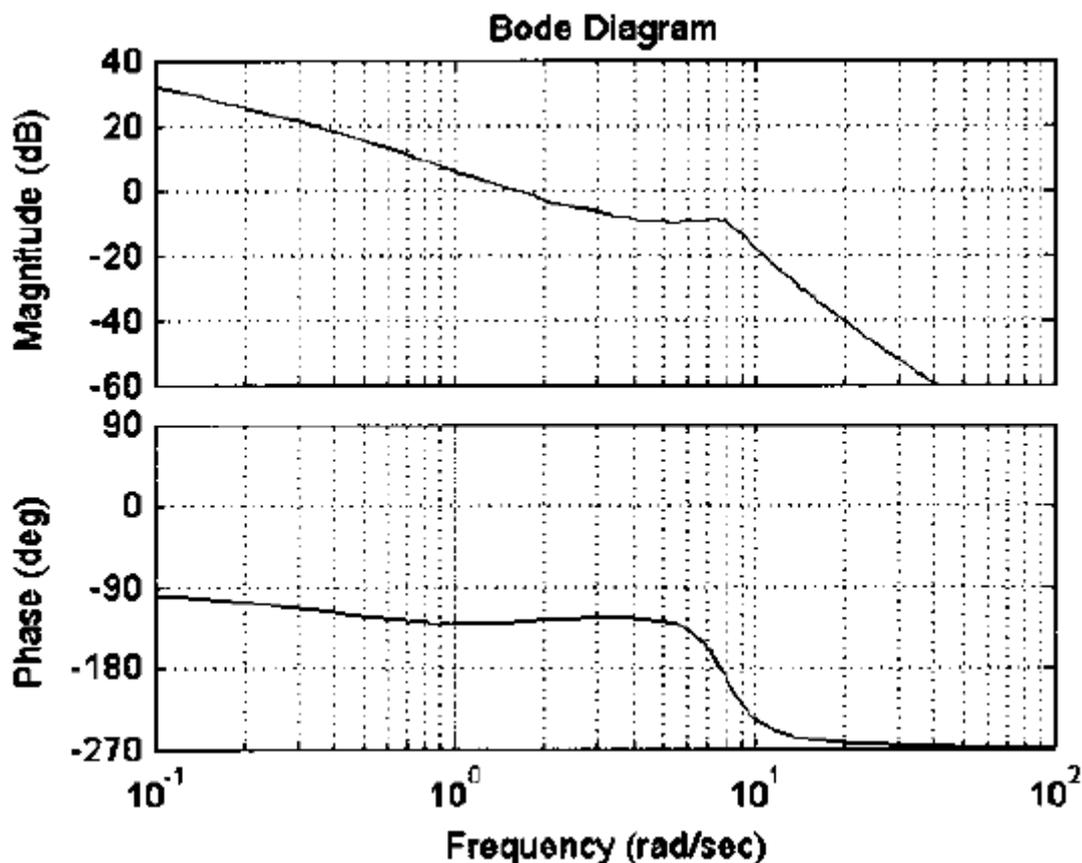


Figure 3.2.1 Magnitude and phase plots of Bode plot

[Source: "Linear Control System Analysis and Design with MATLAB" by John J D'Azzo, Constantine, Stuart, Page: 318]

## BASIC FACTORS OF $G(j\omega)$

The basic factors that are very frequently occur in a typical transfer function  $G(j\omega)$  are,

1. Constant gain,  $K$
2. Integral and derivative factors  $(j\omega)^{\mp 1}$
3. First-order factors  $(1 + j\omega T)^{\mp 1}$
4. Quadratic factors  $(1 + 2\zeta (j \frac{\omega}{\omega_n}) + (\frac{\omega}{\omega_n})^2)^{\mp 1}$

### Constant Gain, $K$

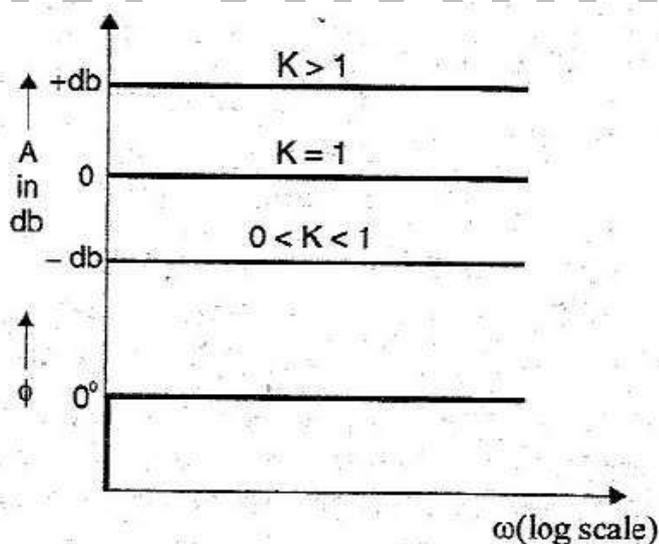
Let  $G(s)=K$ ,

$$G(j\omega) = K = K \angle 0^\circ$$

$$A = |G(j\omega)| \text{ in db} = 20 \log K$$

$$\phi = \angle G(j\omega) = 0^\circ$$

The magnitude plot for a constant gain  $K$  is a horizontal straight line at the magnitude of  $20 \log K$  db. The phase plot is a straight line at  $0^\circ$ .



**Figure 3.2.2 Bode plot of constant gain,  $K$**

[Source: "Control Systems" by A Nagoor Kani, Page: 3.10]

### Integral Factor

Let  $G(s)=K/s$ ,

$$G(j\omega) = \frac{K}{j\omega} = \frac{K}{\omega} \angle -90^\circ$$

$$A = |G(j\omega)| \text{ in db} = 20 \log \left( \frac{K}{\omega} \right)$$

$$\phi = \angle G(j\omega) = -90^\circ$$

The magnitude plot of the integral factor is a straight line with the slope of -20db/dec and passing through zero db when  $\omega=K$ . The phase plot is a straight line at  $-90^\circ$ .

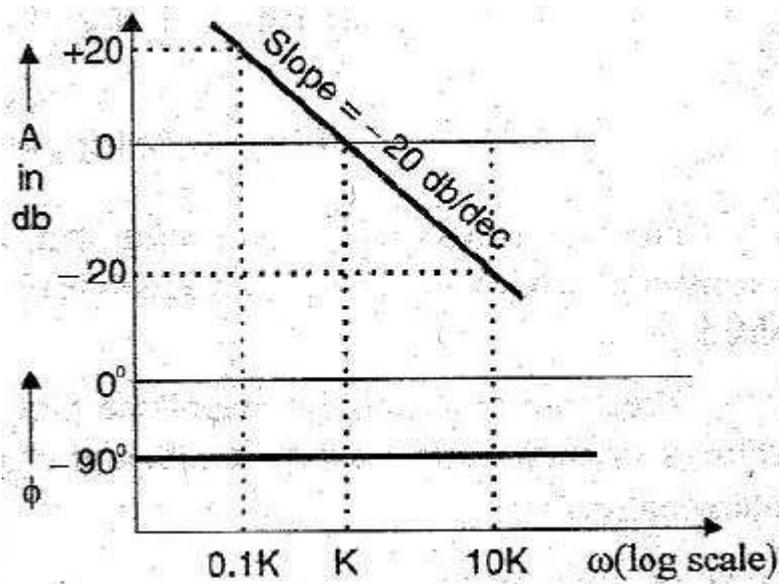


Figure 3.2.3 Bode plot of integral factor,  $K/j\omega$

[Source: "Control Systems" by A Nagoor Kani, Page: 3.11]

### Derivative factor

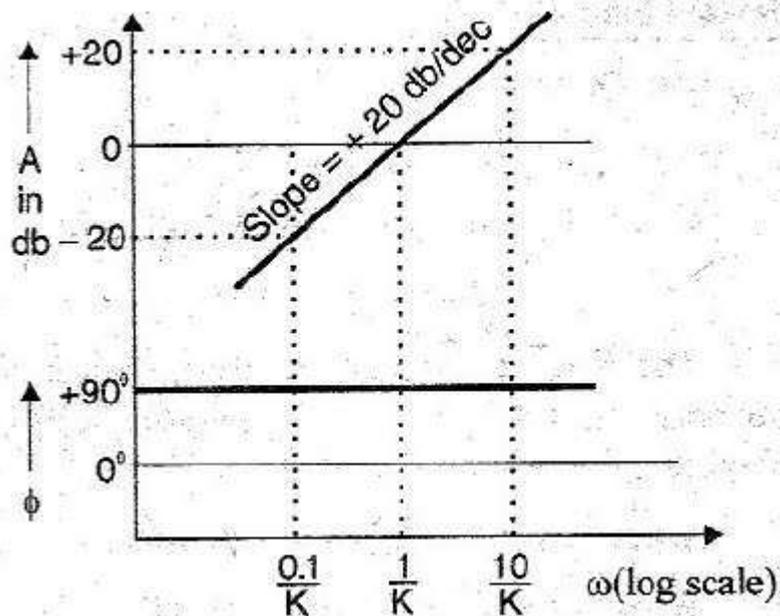
Let  $G(s)=Ks$ ,

$$G(j\omega) = Kj\omega = K\omega \angle 90^\circ$$

$$A = |G(j\omega)| \text{ in db} = 20 \log(K\omega)$$

$$\phi = \angle G(j\omega) = +90^\circ$$

The magnitude plot of the integral factor is a straight line with the slope of 20db/dec and passing through zero db when  $\omega=K$ . The phase plot is a straight line at  $+90^\circ$ .



**Figure 3.2.4 Bode plot of derivative factor,  $K \times j\omega$**

[Source: "Control Systems" by A Nagoor Kani, Page: 3.11]

### First order factor in denominator

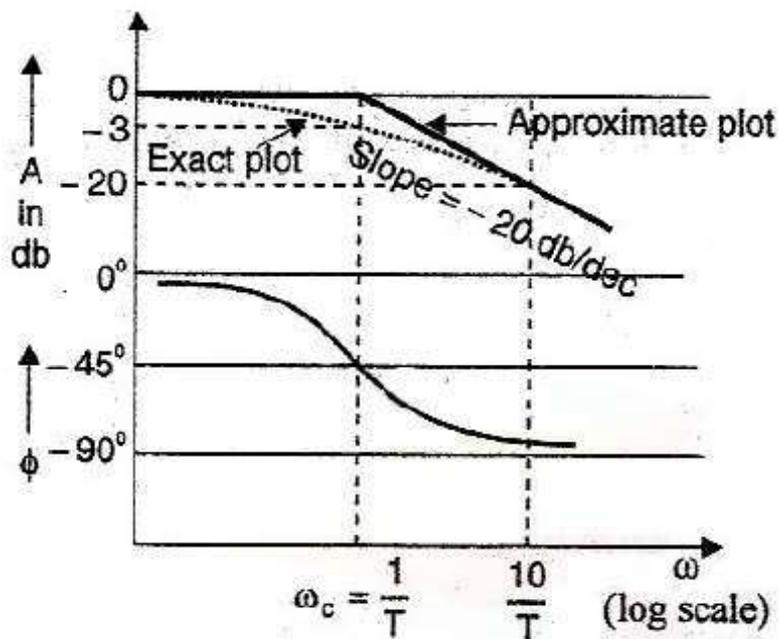
$$\text{Let } G(s) = \frac{1}{1+sT}$$

$$G(j\omega) = \frac{1}{1+j\omega T} = \frac{1}{\sqrt{1+\omega^2 T^2}} \angle -\tan^{-1} \omega T$$

$$A = |G(j\omega)|_{\text{in db}} = 20 \log \left( \frac{1}{\sqrt{1+\omega^2 T^2}} \right)$$

$$\phi = \angle G(j\omega) = \angle -\tan^{-1} \omega T$$

The magnitude plot of the first order factor can be approximated by two straight lines, one is a straight line at zero db for the frequency range,  $0 < \omega < 1/T$ , and the other is a straight line with slope -20db/dec for the frequency range,  $1/T < \omega < \infty$ . The corner frequency is  $\omega_c = 1/T$  and the loss in db at the corner frequency is -3db. The phase angle of the first order factor varies from  $0^\circ$  to  $-90^\circ$  as  $\omega$  is varied from zero to infinity. The phase plot is a curve passing through  $-45^\circ$  at  $\omega_c$ .



**Figure 3.2.5 Bode plot of first order factor in denominator,  $1/(1+j\omega T)$**

[Source: "Control Systems" by A Nagoor Kani, Page: 3.13]

### First order factor in numerator

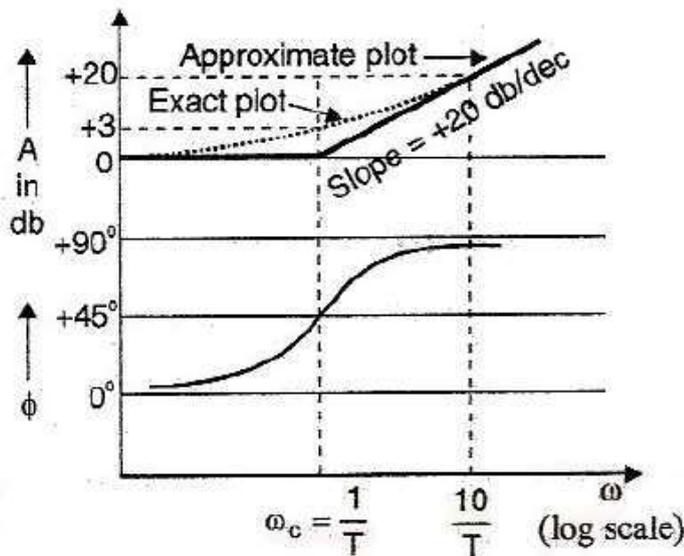
Let  $G(s) = 1 + sT$

$$G(j\omega) = 1 + j\omega T = \sqrt{1 + \omega^2 T^2} \angle \tan^{-1} \omega T$$

$$A = |G(j\omega)| \text{ in db} = 20 \log (\sqrt{1 + \omega^2 T^2})$$

$$\phi = \angle G(j\omega) = \angle \tan^{-1} \omega T$$

The magnitude plot of the first order factor can be approximated by two straight lines, one is a straight line at zero db for the frequency range,  $0 < \omega < 1/T$ , and the other is a straight line with slope 20db/dec for the frequency range,  $1/T < \omega < \infty$ . The corner frequency is  $\omega_c = 1/T$  and the loss in db at the corner frequency is +3db. The phase angle of the first order factor varies from  $0^\circ$  to  $+90^\circ$  as  $\omega$  is varied from zero to infinity. The phase plot is a curve passing through  $+45^\circ$  at  $\omega_c$ .



**Figure 3.2.6 Bode plot of first order factor in numerator,  $(1+j\omega T)$**

[Source: "Control Systems" by A Nagoor Kani, Page: 3.14]

### Quadratic factor in denominator

Second order closed loop transfer function is given by

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{1}{\left(\frac{s}{\omega_n}\right)^2 + 2\zeta\frac{s}{\omega_n} + 1}$$

$$G(j\omega) = \frac{1}{\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\frac{j\omega}{\omega_n} + 1} = \frac{1}{-\left(\frac{\omega}{\omega_n}\right)^2 + 2\zeta\frac{j\omega}{\omega_n} + 1}$$

$$G(j\omega) = \frac{1}{\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + 4\zeta^2\frac{\omega^2}{\omega_n^2}}} \angle -\tan^{-1}\left(\frac{2\zeta\frac{\omega}{\omega_n}}{1 - \frac{\omega^2}{\omega_n^2}}\right)$$

At low frequencies when  $\omega \ll \omega_n$ , the magnitude is,

$$A = -20 \log \sqrt{1 - \frac{\omega^2}{\omega_n^2} (2 - 4\zeta^2) + \frac{\omega^4}{\omega_n^4}} \cong -20 \log 1 = 0$$

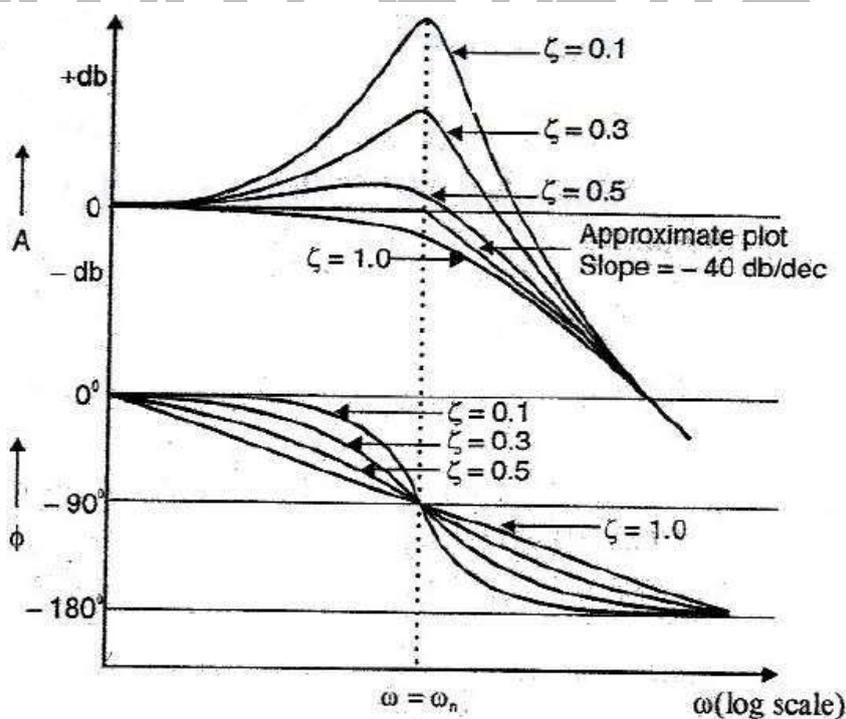
At high frequencies when  $\omega \gg \omega_n$ , the magnitude is,

$$A = -20 \log \sqrt{1 - \frac{\omega^2}{\omega_n^2} (2 - 4\zeta^2) + \frac{\omega^4}{\omega_n^4}}$$

$$A \cong -20 \log \sqrt{\frac{\omega^4}{\omega_n^4}} = -20 \log \frac{\omega^2}{\omega_n^2} = -20 \log \left( \frac{\omega}{\omega_n} \right)^2$$

$$\phi = \angle G(j\omega) = -\tan^{-1} \left( \frac{2\zeta \frac{\omega}{\omega_n}}{1 - \frac{\omega^2}{\omega_n^2}} \right)$$

The magnitude plot of the quadratic factor in the denominator can be approximated by two straight lines, one is a straight line at zero db for the frequency range,  $0 < \omega < \omega_n$ , and the other is a straight line with slope -40db/dec for the frequency range,  $\omega_n < \omega < \infty$ . The frequency at which the two asymptotes meet is called the corner frequency. For the quadratic factor, the frequency,  $\omega_n$  is the corner frequency,  $\omega_c$ . The phase angle of the quadratic factor varies from  $0^\circ$  to  $-180^\circ$  as  $\omega$  is varied from zero to infinity. The phase plot is a curve passing through  $-90^\circ$  at  $\omega_c$ . At the corner frequency, phase angle is  $-90^\circ$  and independent of  $\zeta$ , but at all other frequency it depends on  $\zeta$ .



**Figure 3.2.7 Bode plot of quadratic factor in denominator**

[Source: "Control Systems" by A Nagoor Kani, Page: 3.15]

### Quadratic factor in the numerator

$$G(s) = \frac{s^2 + 2\zeta\omega_n s + \omega_n^2}{\omega_n^2} = \left(\frac{s}{\omega_n}\right)^2 + 2\zeta\frac{s}{\omega_n} + 1$$

$$G(j\omega) = \left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\frac{j\omega}{\omega_n} + 1 = -\left(\frac{\omega}{\omega_n}\right)^2 + j\frac{2\zeta\omega}{\omega_n} + 1$$

$$|G(j\omega)| = \sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + 4\zeta^2\frac{\omega^2}{\omega_n^2}} \quad \angle \tan^{-1}\left(\frac{2\zeta\frac{\omega}{\omega_n}}{1 - \frac{\omega^2}{\omega_n^2}}\right)$$

At low frequencies when  $\omega \ll \omega_n$ , the magnitude is,

$$A = 20 \log \sqrt{1 - \frac{\omega^2}{\omega_n^2} (2 - 4\zeta^2) + \frac{\omega^4}{\omega_n^4}} \cong 20 \log 1 = 0$$

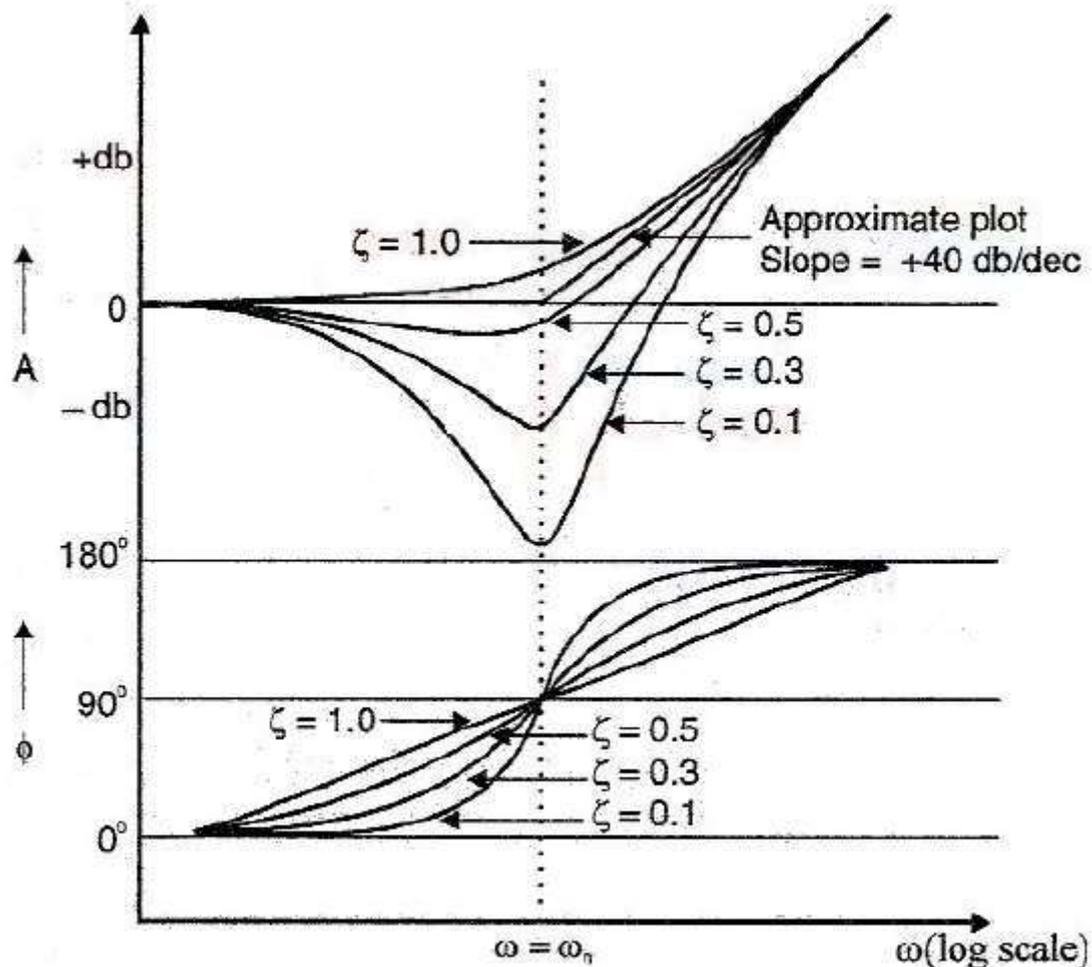
At high frequencies when  $\omega \gg \omega_n$ , the magnitude is,

$$A = 20 \log \sqrt{1 - \frac{\omega^2}{\omega_n^2} (2 - 4\zeta^2) + \frac{\omega^4}{\omega_n^4}}$$

$$A \cong 20 \log \sqrt{\frac{\omega^4}{\omega_n^4}} = 20 \log \frac{\omega^2}{\omega_n^2} = 20 \log \left(\frac{\omega}{\omega_n}\right)^2$$

$$\phi = \angle G(j\omega) = \tan^{-1} \left( \frac{2\zeta\frac{\omega}{\omega_n}}{1 - \frac{\omega^2}{\omega_n^2}} \right)$$

The magnitude plot of the quadratic factor in the denominator can be approximated by two straight lines, one is a straight line at zero db for the frequency range,  $0 < \omega < \omega_n$ , and the other is a straight line with slope +40db/dec for the frequency range,  $\omega_n < \omega < \infty$ . The frequency at which the two asymptotes meet is called the corner frequency. For the quadratic factor, the frequency,  $\omega_n$  is the corner frequency,  $\omega_c$ . The phase angle of the quadratic factor varies from  $0^\circ$  to  $+180^\circ$  as  $\omega$  is varied from zero to infinity. The phase plot is a curve passing through  $+90^\circ$  at  $\omega_c$ . At the corner frequency, phase angle is  $+90^\circ$  and independent of  $\zeta$ , but at all other frequency it depends on  $\zeta$ .



**Figure 3.2.8 Bode plot of quadratic factor in numerator**

[Source: "Control Systems" by A Nagoor Kani, Page: 3.16]

**Derivative factor: magnitude**

$$M = 20 \log |j\omega| = 20 \log \omega \text{ dB}$$

$$\angle j\omega = 90^\circ$$

$$\Delta M = 20 \log \omega_2 - 20 \log \omega_1 = 20 \log \frac{\omega_2}{\omega_1} \text{ dB/decade}$$

$$\Delta M = 20 \log 10 = 20 \text{ dB/decade}$$

$$\Delta M = 20 \log 2 \approx 6 \text{ dB/octave}$$

**Integral factor: magnitude**

$$M = 20 \log \left| \frac{1}{j\omega} \right| = -20 \log \omega \text{ dB}$$

$$\angle j\omega = 270^\circ$$

$$\Delta M = -20 \log \omega_2 + 20 \log \omega_1 = -20 \log \frac{\omega_2}{\omega_1} \text{ dB/decade}$$

$$\Delta M = -20 \log 10 = -20 \text{ dB/decade}$$

$$\Delta M = 20 \log 2 \approx -6 \text{ dB/octave}$$

**First-order derivative factor: magnitude**

$$M = 20 \log |1 + j\omega\tau| = 20 \log (\sqrt{1 + [\omega\tau]^2}) \text{ dB}$$

For  $\omega \ll \omega_c$ ,  $M \approx 0 \text{ dB}$

For  $\omega \gg \omega_c$

$$M \approx 20 \log \frac{\omega}{\omega_c} \text{ dB}$$

Here,  $\omega_c = 1/\tau = \text{corner frequency}$

For  $\omega > \omega_c$

$$\Delta M = 20 \log \omega_2 - 20 \log \omega_1 = 20 \log \frac{\omega_2}{\omega_1}$$

$$\Delta M = 20 \log 10 = 20 \text{ dB/decade}$$

$$\Delta M = 20 \log 2 \approx 6 \text{ dB/octave}$$

**First-order derivative factor: phase**

$$\theta = \angle 1 + j\omega\tau = \arctan(\omega\tau)$$

$$\theta \approx 0 \quad ; w < \frac{w_c}{10}$$

$$\theta = 45^\circ \left( 1 + \log \frac{\omega}{\omega_c} \right) \quad ; \frac{w_c}{10} < w < 10w_c$$

$$\theta \approx 90 \quad ; w > 10w_c$$

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**First-order integral factor: magnitude**

$$M = 20 \log \left| \frac{1}{1 + j\omega\tau} \right| = 20 \log \left( \frac{1}{\sqrt{1 + [\omega\tau]^2}} \right) \text{dB}$$

$$M \approx 0, \quad \omega \ll \omega_c$$

$$M \approx -20 \log \frac{\omega}{\omega_c} \text{dB}, \quad \omega \gg \omega_c$$

$$\Delta M = -20 \log \omega_2 + 20 \log \omega_1 = -20 \log \frac{\omega_2}{\omega_1} \text{dB/decade}$$

$$\Delta M = -20 \log 2 \approx -6 \text{ dB/octave}$$

**First-order integral factor: phase**

$$\theta = 360, \quad \omega < \omega_c / 10$$

$$\theta = 360 - 45^\circ \left( 1 + \log \frac{\omega}{\omega_c} \right), \quad \omega_c / 10 < \omega < 10 \omega_c$$

$$\theta = 360 - 45^\circ \left( 1 + \log \frac{\omega}{\omega_c} \right)$$

$$\theta = 270, \quad \omega > 10 \omega_c$$

**Second-order derivative factor: magnitude**

$$M = 20 \log | \omega_n^2 - \omega^2 + j2\zeta\omega\omega_n |$$
$$= 20 \log \left( \omega_n^2 \sqrt{\left( 1 - \frac{\omega^2}{\omega_n^2} \right)^2 + \left( 2\zeta \frac{\omega}{\omega_n} \right)^2} \right)$$

$$M \approx 40 \log \omega_n, \quad \omega \ll \omega_n$$

$$M = 20 \log (2\zeta \omega_n^2), \quad \omega = \omega_n$$

$$M = 40 \log \omega, \quad \omega \gg \omega_n$$

For  $\omega \gg \omega_n$

$$\Delta M = 40 \log \omega_2 - 40 \log \omega_1 = 40 \log \frac{\omega_2}{\omega_1} \text{ dB/decade}$$

$$\Delta M = 40 \log 10 = 40 \text{ dB/decade}$$

$$\Delta M = 40 \log 2 \approx 12 \text{ dB/octave}$$

**Second-order derivative factor: phase**

$$\theta = \angle | \omega_n^2 - \omega^2 + j2\zeta\omega\omega_n | = \arctan \left( \frac{2\zeta \frac{\omega}{\omega_n}}{1 - \frac{\omega^2}{\omega_n^2}} \right)$$

$$\theta = 0^\circ, \quad \omega < \frac{\omega_n}{10}$$

$$\theta = 90^\circ, \quad \omega = \omega_n$$

$$\theta = 180^\circ, \quad \omega > 10\omega_n$$

**Second-order integral factor**

$$M = 20 \log \left| \frac{1}{\omega_n^2 - \omega^2 + j2\zeta\omega\omega_n} \right| \text{ dB} = 20 \log \left( \frac{1}{\omega_n^2 \sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta \frac{\omega}{\omega_n}\right)^2}} \right) \text{ dB}$$

$$M \approx -40 \log \omega_n, \quad \omega \ll \omega_n$$

$$M = -20 \log (2\zeta\omega_n^2), \quad \omega = \omega_n$$

$$M = -40 \log \omega, \quad \omega \gg \omega_n$$

$$\Delta M = -40 \log \omega_2 + 40 \log \omega_1 = -40 \log \frac{\omega_2}{\omega_1} \text{ dB / decade}$$

**PROCEDURE FOR MAGNITUDE PLOT OF BODE PLOT**

**Step 1:** Convert the transfer function into Bode form or time constant form.

**Step 2:** List the corner frequencies in the increasing order and prepare a table as shown

Term	Corner frequency rad/sec	Slope db/dec	Change in Slope db/dec

In the above table, enter  $K$  or  $K/(j\omega)^n$  or  $K(j\omega)^n$  as the first term and the other terms in the increasing order of corner frequencies. Then enter the corner frequency, slope contributed by each term and change in slope at every corner frequency.

**Step 3:** Choose an arbitrary frequency  $\omega_1$  which is lesser than the lowest corner frequency. Calculate the db magnitude of  $K$  or  $K/(j\omega)^n$  or  $K(j\omega)^n$  at  $\omega_1$  and at the lowest corner frequency.

**Step 4:** Then calculate the gain (db magnitude) at every corner frequency one by one by using the formula,

$$\text{Gain at } \omega_y = \text{change in gain from } \omega_x \text{ to } \omega_y + \text{Gain at } \omega_x$$

$$A_y = (\text{Slope from } \omega_x \text{ to } \omega_y \times \log(\omega_y / \omega_x)) + \text{Gain at } \omega_x$$

**Step 5:** Choose an arbitrary frequency  $\omega_h$  which is greater than the highest corner frequency. Calculate the gain at  $\omega_h$  by using the formula in step 4.

**Step 6:** In a semilog graph sheet mark the required range of frequency on x-axis (log scale) and the range of db magnitude on y-axis (ordinary scale) after choosing proper units.

**Step 7:** Mark all the points obtained in steps 3, 4, 5 on the graph and join the points by straight lines. Mark the slope at every part of the graph.

#### PROCEDURE FOR PHASE PLOT OF BODE PLOT

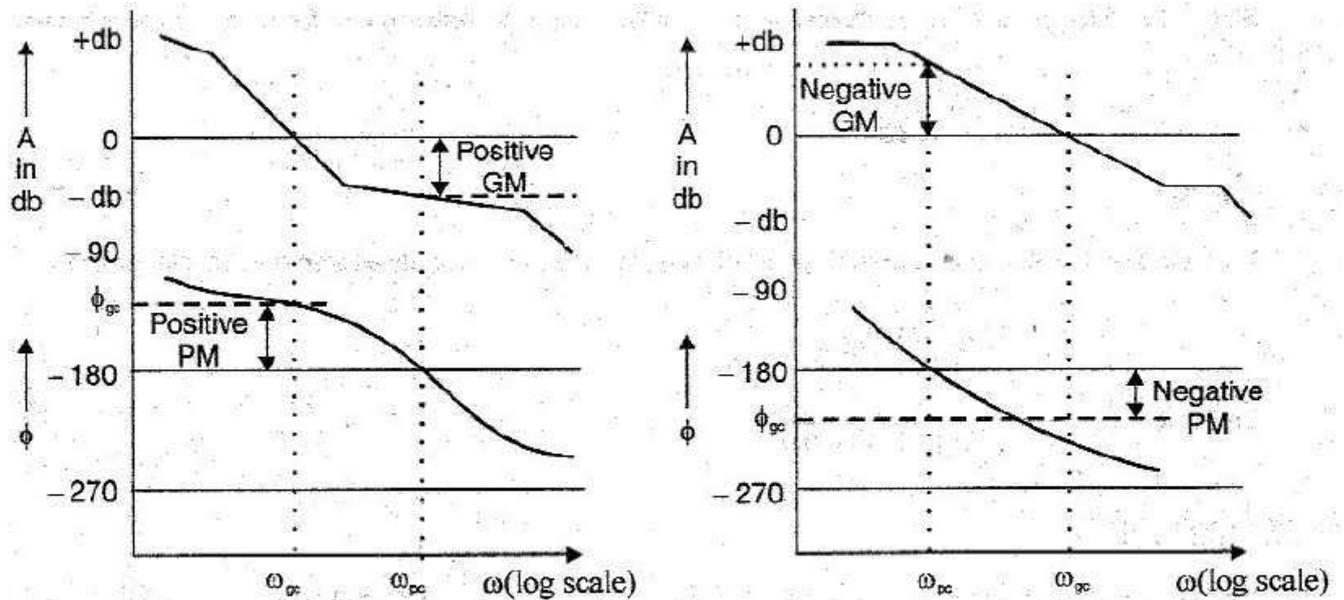
The phase plot is an exact plot obtained with exact phase angles of  $G(j\omega)$  computed for various values of  $\omega$  and is then tabulated. The choice of frequencies are preferably the frequencies chosen for magnitude plot. Usually the magnitude plot and phase plot are drawn in a single semilog sheet on a common frequency scale. Take another y-axis in the graph where the magnitude plot is drawn and, in this y-axis, mark the desired range of phase angles after choosing proper units. From the tabulated values of  $\omega$  and phase angles, mark all the points on the graph. Join the points by a smooth curve.

#### DETERMINATION OF GAIN MARGIN AND PHASE MARGIN FROM BODE PLOT

The gain margin in db is given by the negative of db magnitude of  $G(j\omega)$  at the phase crossover frequency,  $\omega_{pc}$ . The  $\omega_{pc}$  is the frequency at which phase of  $G(j\omega)$  is  $180^\circ$ . If the db magnitude of  $G(j\omega)$  at  $\omega_{pc}$  is negative then gain margin is positive and vice versa.

Let  $\Phi_{gc}$  be the phase angle of  $G(j\omega)$  at gain cross over frequency,  $\omega_{gc}$ . The  $\omega_{gc}$  is the frequency at which the db magnitude of  $G(j\omega)$  is zero. Now the phase margin,  $\gamma$  is

given by,  $\gamma = 180^\circ + \Phi_{gc}$ . If  $\Phi_{gc}$  is less negative than  $-180^\circ$ , then phase margin is positive and vice versa. The positive and negative gain margins and phase margins are illustrated in figure 3.2.9.



**Figure 3.2.9 Gain margin and Phase margin in Bode plot**

[Source: "Control Systems" by A Nagoor Kani, Page: 3.20]

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### 3.5 CORRELATION BETWEEN FREQUENCY DOMAIN AND TIME DOMAIN SPECIFICATIONS

For a second order system,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Put  $s=j\omega$

$$\frac{C(j\omega)}{R(j\omega)} = \frac{\omega_n^2}{-\omega^2 + 2\zeta\omega_n j\omega + \omega_n^2}$$

$$\frac{C(j\omega)}{R(j\omega)} = \frac{1}{\omega^2 - \frac{\omega_n^2}{\omega} + 2\zeta j\omega_n + 1}$$

Let  $u = \frac{\omega}{\omega_n}$ , then

$$\frac{C(j\omega)}{R(j\omega)} = \frac{1}{(1 - u^2) + 2\zeta ju}$$

We know,

$$M(j\omega) = |M(j\omega)| \angle M(j\omega)$$

$$|M(j\omega)| = \frac{1}{\sqrt{(1 - u^2)^2 + (2\zeta u)^2}}$$

$$\theta = -\tan^{-1} \left( \frac{2\zeta u}{1 - u^2} \right)$$

Now,

$$M_r = \frac{1}{2\zeta\sqrt{1 - \zeta^2}}$$

$$\omega_r = \omega_n \sqrt{1 - 2\zeta^2}$$

$$\omega_b = \omega_n \sqrt{1 - 2\zeta^2 + \sqrt{4\zeta^4 - 4\zeta^2 + 2}}$$

$$PM = -180^\circ + \phi$$

where,

$$\phi = \tan^{-1} \frac{\text{_____}}{\sqrt{4\zeta^2 + 1 - 2\zeta^2}}$$

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### 3.4 DETERMINATION OF CLOSED LOOP RESPONSE FROM OPEN LOOP RESPONSE

#### M and N circles

Peak magnitude

$$M_r = 20 \log \left| \frac{C(j\omega)}{R(j\omega)} \right| \text{ dB}$$

where, 3 dB is considered good.

#### M-CIRCLES

$$M(j\omega) = \frac{G(j\omega)}{1 + G(j\omega)}$$

$$G(j\omega) = X + jY$$

$$M(j\omega) = \frac{X + jY}{1 + X + jY} = \frac{\sqrt{X^2 + Y^2} \angle \tan^{-1} \frac{Y}{X}}{\sqrt{(1+X)^2 + Y^2} \angle \tan^{-1} \left( \frac{Y}{1+X} \right)}$$

$$= \frac{\sqrt{X^2 + Y^2}}{\sqrt{(1+X)^2 + Y^2}} \angle \tan^{-1} \left( \frac{Y}{X} \right) - \tan^{-1} \left( \frac{Y}{1+X} \right)$$

Let, M = Magnitude of M(jω)

$$|M(j\omega)| = \frac{\sqrt{X^2 + Y^2}}{\sqrt{(1+X)^2 + Y^2}}$$

$$M^2(1+X)^2 + M^2Y^2 = X^2 + Y^2$$

$$X^2(1 - M^2) + (1 - M^2)Y^2 - 2M^2X = M^2$$

$$X^2 + Y^2 - 2 \frac{M^2}{(1 - M^2)} X = \frac{M^2}{(1 - M^2)}$$

Adding  $\left(\frac{M}{1-M^2}\right)^2$  on both sides, we get,

(1-M)

$$\left(X - \frac{M}{1-M^2}\right)^2 + Y^2 = \left(\frac{M}{1-M^2}\right)^2$$

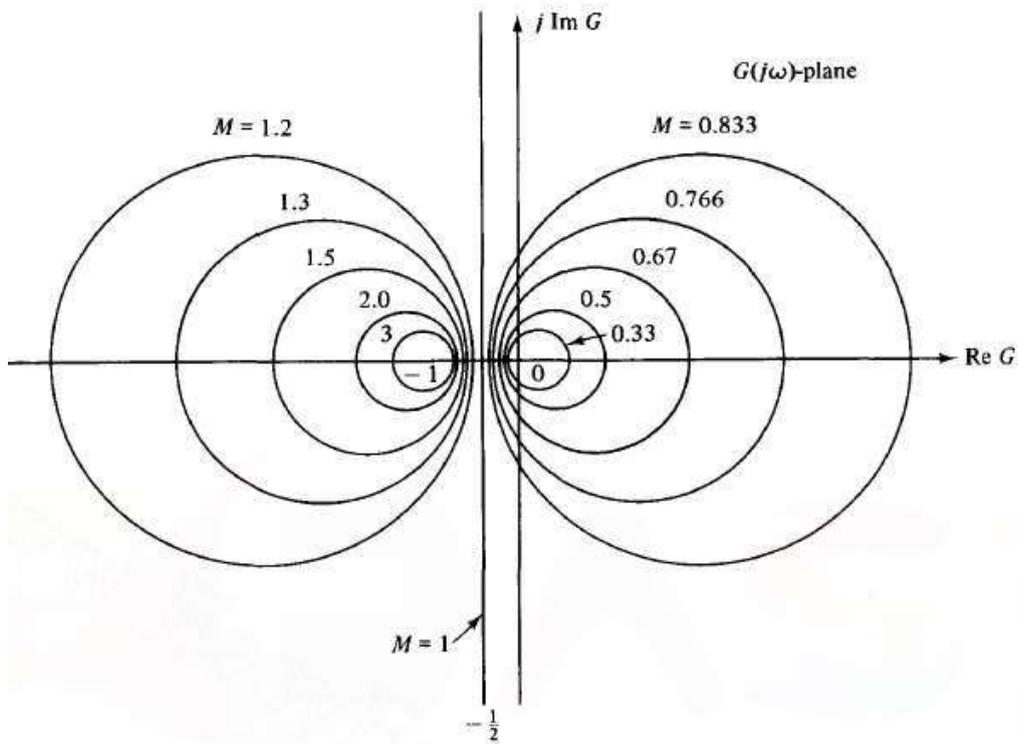
(1 - M<sup>2</sup>)

(1 - M<sup>2</sup>)

centre at  $\left(\frac{M^2}{1-M^2}, 0\right)$  and radius  $\frac{M}{1-M^2}$

Family of M-circles corresponding to the closed loop magnitudes, M of a unit feedback system is given by the figure 3.4.1.

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**Figure 3.4.1 Constant M-circles in the polar co-ordinates**

[Source: "Automatic Control Systems" by Benjamin C. Kuo, Page: 487]

**N-CIRCLES**

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$$\angle M(j\omega) = \alpha = \frac{\angle G(j\omega)}{\angle(1 + G(j\omega))}$$

$$\alpha = \tan^{-1} \frac{Y}{X} - \tan^{-1} \frac{Y}{1 + X}$$

$$\tan \alpha = N = \tan \left( \tan^{-1} \frac{Y}{X} - \tan^{-1} \frac{Y}{1 + X} \right)$$

We know,

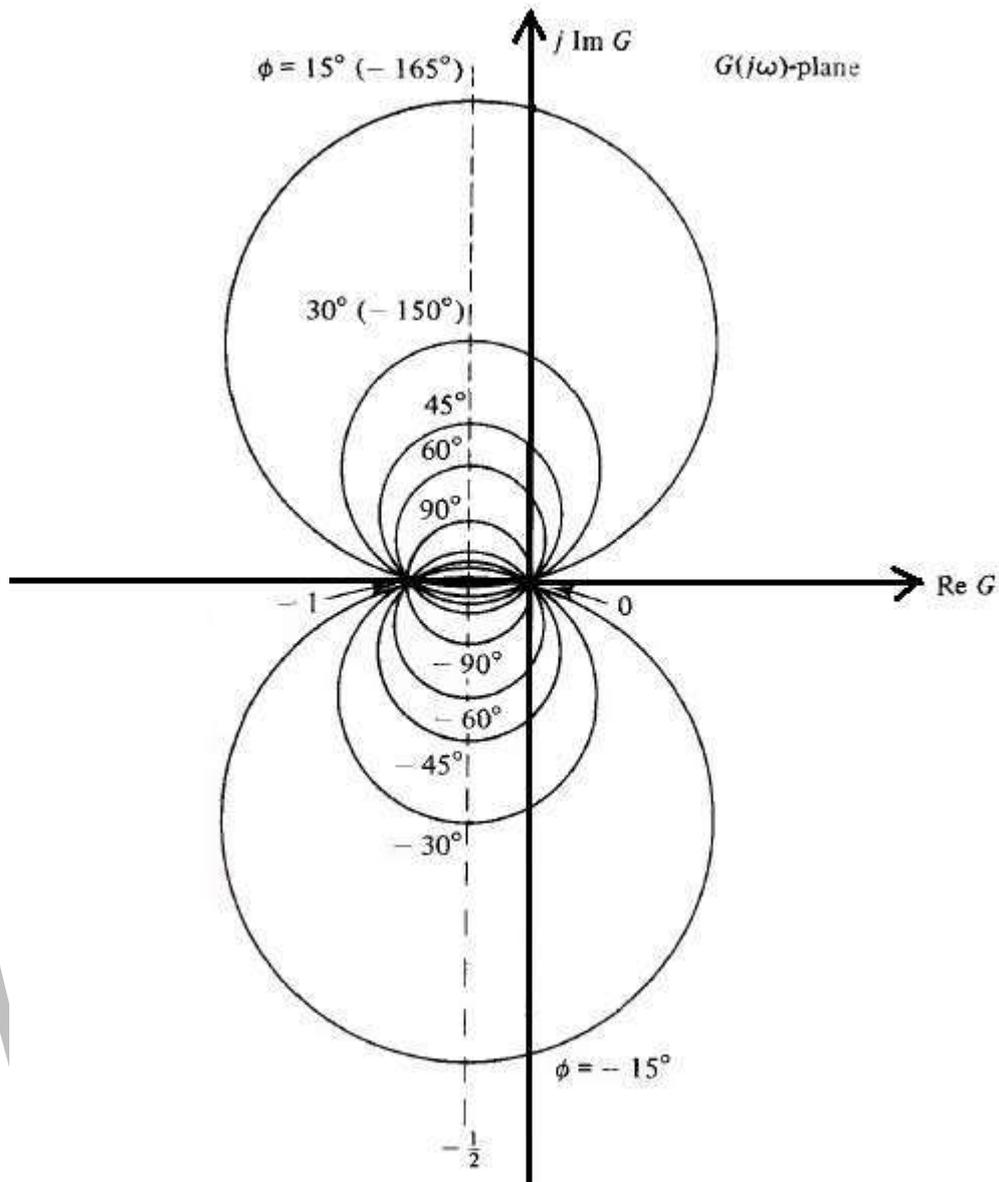
$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

$$N = \left( \frac{Y}{X^2 + X + Y^2} \right)$$

$$\left( X + \frac{1}{2} \right)^2 + \left( Y - \frac{1}{2N} \right)^2 = \frac{1}{4} + \left( \frac{1}{2N} \right)^2$$

The above equation represents the family of circles with its

Centre at  $\left( -\frac{1}{2}, \frac{1}{2N} \right)$  and radius  $\sqrt{\frac{1}{4} + \left( \frac{1}{2N} \right)^2}$



**Figure 3.4.2 Constant N-circles in the polar co-ordinates**

[Source: "Automatic Control Systems" by Benjamin C. Kuo, Page: 490]

### 3.1 FREQUENCY RESPONSE

The response of a system for the sinusoidal input is called sinusoidal response. The ratio of sinusoidal response to sinusoidal input is called sinusoidal transfer function of the system and in general, it is denoted by,  $T(j\omega)$ . The sinusoidal transfer function is the frequency domain representation of the system and so it is also called frequency domain transfer function.

The frequency domain transfer function  $T(j\omega)$  is a complex function of  $\omega$ . Hence, it can be separated into magnitude function and phase function. Now, the magnitude and phase functions will be real functions of  $\omega$  and they are called frequency response.

The frequency response can be evaluated for open loop system and closed loop system. The frequency domain transfer function of open loop and closed loop systems can be obtained from the s-domain transfer function by replacing 's' by  $j\omega$  as shown:

**Open loop transfer function:**  $G(j\omega) = |G(j\omega)|\angle G(j\omega)$

**Loop transfer function:**  $G(j\omega)H(j\omega) = |G(j\omega)H(j\omega)|\angle G(j\omega)H(j\omega)$

**Closed loop transfer function:**  $M(j\omega) = |M(j\omega)|\angle M(j\omega)$

The advantages of frequency response analysis are the following:

1. The absolute and relative stability of the closed loop system can be estimated from the knowledge of their open loop frequency response.
2. The practical testing of systems can be easily carried with available sinusoidal signal generators and precise measurement equipments.
3. The transfer function of complicated systems can be determined experimentally by frequency response tests.
4. The design and parameter adjustment of the open loop transfer function of a system for specified closed loop performance is carried out more easily in frequency domain.
5. When the system is designed by the use of frequency response analysis, the effects of noise disturbance and parameter variations are relatively easy to visualize and incorporate corrective measures.
6. The frequency response analysis and designs can be extended to certain non-linear control systems.

The frequency response of a system is a frequency dependent function which expresses how a sinusoidal signal of a given frequency on the system input is transferred through the system. Time-varying signals at least periodical signals – which excite systems, as the reference (set point) signal or a disturbance in a control system or measurement signals which are inputs signals to signal filters, can be regarded as consisting of a sum of frequency components. Each frequency component is a sinusoidal signal having certain amplitude and a certain frequency. (The Fourier series expansion or the Fourier transform can be used to express these frequency components quantitatively.) The frequency response expresses how each of these frequency components is transferred through the system. Some components may be amplified, others may be attenuated, and there will be some phase lag through the system. The frequency response is an important tool for analysis and design of signal filters (as low pass filters and high pass filters), and for analysis, and to some extent, design, of control systems. Both signal filtering and control systems applications are described (briefly) later in this chapter. The definition of the frequency response – which will be given in the next section – applies only to linear models, but this linear model may very well be the local linear model about some operating point of a non-linear model. The frequency response can be found experimentally or from a transfer function model. It can be presented graphically or as a mathematical function.

## **FREQUENCY DOMAIN SPECIFICATIONS**

The performance and characteristics of a system in frequency domain are measured in terms of frequency domain specifications. The requirements of a system to be designed are usually specified in terms of these specifications.

***The frequency domain specifications are,***

- a) Resonant peak,  $M_r$
- b) Resonant frequency,  $\omega_r$
- c) Bandwidth,  $\omega_b$
- d) Cut-off rate
- e) Gain margin,  $K_g$
- f) Phase margin,  $\gamma$

## FREQUENCY DOMAIN SPECIFICATIONS OF SECOND ORDER SYSTEM

### Resonant peak, $M_r$

The maximum value of the magnitude of closed loop transfer function is called the resonant peak,  $M_r$ . A large resonant peak corresponds to a large overshoot in transient response. Consider the closed loop transfer function of second order system,

$$\frac{C(s)}{R(s)} = M(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

The sinusoidal transfer function  $M(j\omega)$  is obtained by letting  $s=j\omega$ .

$$\begin{aligned} M(j\omega) &= \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2} \\ &= \frac{\omega_n^2}{-\omega^2 + 2j\zeta\omega_n(j\omega) + \omega_n^2} \\ &= \frac{\omega_n^2}{\omega_n^2 \left( -\frac{\omega^2}{\omega_n^2} + 2j\zeta\frac{\omega}{\omega_n} + 1 \right)} \\ &= \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2 + 2j\zeta\frac{\omega}{\omega_n}} \end{aligned}$$

Let normalized frequency,  $u = \left(\frac{\omega}{\omega_n}\right)$ ,

$$M(j\omega) = \frac{1}{1 - u^2 + 2j\zeta u}$$

Let,  $M$  – Magnitude of closed loop transfer function  
 $\alpha$  – Phase of closed loop transfer function

$$M = |M(j\omega)| = \left[ (1 - u^2)^2 + 4\zeta^2 u^2 \right]^{-\frac{1}{2}}$$

$$\alpha = \angle M(j\omega) = -\tan^{-1} \frac{2\zeta u}{1 - u^2}$$

The resonant peak is the maximum value of  $M$ . The condition for maximum value of  $M$  can be obtained by differentiating the equation of  $M$  with respect to  $u$  and letting  $(dM/du=0)$  when  $(u=u_r)$  with normalized frequency,  $u_r = \frac{\omega_r}{\omega_n}$

On differentiating 'M' with respect to 'u', we get,

$$\frac{dM}{du} = \frac{d}{du} [1 - u^2 + 2j\zeta u]^{\frac{1}{2}}$$

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$$\begin{aligned}
 &= -\frac{1}{2} [1 - u^2 + 2j\zeta u]^{-\frac{3}{2}} [2(1 - u^2)(-2u) + 8\zeta^2 u] \\
 &= -\frac{[-4u(1 - u^2) + 8\zeta^2 u]}{2[(1 - u^2)^2 + 4\zeta^2 u^2]^{\frac{3}{2}}} \\
 &= -\frac{[4u(1 - u^2) - 8\zeta^2 u]}{2[(1 - u^2)^2 + 4\zeta^2 u^2]^{\frac{3}{2}}}
 \end{aligned}$$

Replacing u by  $u_r$  and equating  $dM/du$  to zero,

$$\frac{[4u_r(1 - u_r^2) - 8\zeta^2 u_r]}{2[(1 - u_r^2)^2 + 4\zeta^2 u_r^2]^{\frac{3}{2}}} = 0$$

$$4u_r(1 - u_r^2) - 8\zeta^2 u_r = 0$$

$$4u_r - 4u_r^3 - 8\zeta^2 u_r = 0$$

$$4u_r - 4u_r^3 = 8\zeta^2 u_r$$

$$4u_r^3 = 4u_r - 8\zeta^2 u_r$$

$$u_r^2 = 1 - 2\zeta^2$$

$$u_r = \sqrt{1 - 2\zeta^2}$$

Therefore, the resonant peak occurs when  $u_r = \sqrt{1 - 2\zeta^2}$

On substituting for M with  $M=M_r$  and  $u=u_r$ ,

$$\begin{aligned}
 M_r &= \frac{1}{[(1 - u_r^2)^2 + 4\zeta^2 u_r^2]^{\frac{1}{2}}} = \frac{1}{[(1 - (1 - 2\zeta^2))^2 + 4\zeta^2(1 - 2\zeta^2)]^{\frac{1}{2}}} \\
 &= \frac{1}{[4\zeta^4 + 4\zeta^2 - 8\zeta^4]^{\frac{1}{2}}} = \frac{1}{[4\zeta^2 - 4\zeta^4]^{\frac{1}{2}}} = \frac{1}{[4\zeta^2(1 - \zeta^2)]^{\frac{1}{2}}} \\
 M_r &= \frac{1}{2\zeta\sqrt{1 - \zeta^2}}
 \end{aligned}$$

### Resonant frequency, $\omega_r$

The frequency at which the resonant peak occurs is called resonant frequency,  $\omega_r$ . This is related to the frequency of oscillation in the step response and thus it is indicative

of the speed of transient response.

Normalized resonant frequency,

$$u_r = \frac{\omega}{\omega_n} = \sqrt{1 - 2\zeta^2}$$

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$$\omega_r = \omega_n \sqrt{1 - 2\zeta^2}$$

### Bandwidth, $\omega_b$

The bandwidth is the range of frequencies for which the system normalized gain is more than -3db. The frequency at which the gain is -3db is called cut-off frequency. Bandwidth is usually defined for closed loop system and it transmits the signals whose frequencies are less than the cut-off frequency. The bandwidth is a measure of the ability of a feedback system to reproduce the input signal, noise rejection characteristics and rise time. A large bandwidth corresponds to a small rise time or fast response.

Let, normalized bandwidth,

$$u_b = \frac{\omega_b}{\omega_n}$$

When  $u=u_b$ , the magnitude  $M$ , of the closed loop system is  $1/\sqrt{2}$  or (-3db) On

substituting for  $M$  with  $u=u_b$  and equating it to  $1/\sqrt{2}$

$$M = \frac{1}{\sqrt{[(1 - u_b^2)^2 + 4\zeta^2 u_b^2]}} = \frac{1}{\sqrt{2}}$$

On squaring and cross multiplying, we get,

$$\begin{aligned} (1 - u_b^2)^2 + 4\zeta^2 u_b^2 &= 2 \\ 1 + u_b^4 - 2u_b^2 + 4\zeta^2 u_b^2 &= 2 \\ u_b^4 - 2u_b^2(1 - 2\zeta^2) - 1 &= 0 \end{aligned}$$

Let  $x = u_b^2$ ,

$$x^2 - 2x(1 - 2\zeta^2) - 1 = 0$$

Hence,

$$x = \frac{2(1 - 2\zeta^2) \pm \sqrt{4(1 - 2\zeta^2)^2 + 4}}{2} = \frac{2(1 - 2\zeta^2) \pm 2\sqrt{(1 - 2\zeta^2)^2 + 1}}{2}$$

Let us take only the positive sign,

$$x = 1 - 2\zeta^2 + \sqrt{(1 - 2\zeta^2)^2 + 1}$$

But,  $u_b = \sqrt{x}$

$$u_b = \sqrt{1 - 2\zeta^2 + \sqrt{(1 - 2\zeta^2)^2 + 1}}$$

Also,  $u_b = \frac{\omega_b}{\omega_n}$

$$\omega_b = \omega_n [1 - 2\zeta^2 + \sqrt{(2 + 4\zeta^4 - 4\zeta^2)}]^{1/2}$$

### Cut-off rate

The slope of the log-magnitude curve near the cut-off frequency is called cut-off rate. The cut-off rate indicates the ability of the system to distinguish the signal from noise.

### Gain margin, $K_g$

The gain margin,  $K_g$  is defined as the value of gain, to be added to system, in order to bring the system to the verge of instability. The gain margin is given by the reciprocal of the magnitude of open loop transfer function at phase crossover frequency.

The frequency at which the phase of open loop transfer function is  $180^\circ$  is called the phase crossover frequency,  $\omega_{pc}$ .

$$K_g = \frac{1}{|G(j\omega_{pc})|}$$

$$K_g \text{ in db} = 20 \log K_g = 20 \log \frac{1}{|G(j\omega_{pc})|}$$

The gain margin in db is given by the negative of the db magnitude of  $G(j\omega)$  at phase crossover frequency. The gain margin indicates the additional gain that can be provided to system without affecting the stability of the system.

[Note: The gain margin of second order system is infinite].

### Phase margin, $\gamma$

The phase margin,  $\gamma$  is defined as the additional phase lag to be added at the gain crossover frequency in order to bring the system to the verge of instability.

The gain crossover frequency,  $\omega_{gc}$  is the frequency at which the magnitude of the open loop transfer function is unity (or it is the frequency at which the db magnitude is zero).

The phase margin is obtained by adding  $180^\circ$  to the phase angle,  $\phi$  of the open loop transfer function at the gain crossover frequency. The phase margin indicates the additional phase lag that can be provided to the system without affecting stability.

$$G(s) = \frac{\omega_n^2}{s(s + 2\zeta\omega_n)}$$

Put  $s=j\omega$ ,

$$G(j\omega) = \frac{\omega^2}{j\omega(j\omega + 2\zeta\omega_n)^n} = \frac{\omega^2}{\omega_n(j\frac{\omega}{\omega_n})\omega_n(j\frac{\omega}{\omega_n} + 2\zeta)^n} = \frac{1}{(j\frac{\omega}{\omega_n})(j\frac{\omega}{\omega_n} + 2\zeta)^n}$$

Let normalized frequency,  $u = \frac{\omega}{\omega_n}$

$$G(j\omega) = \frac{1}{(ju)(ju + 2\zeta)}$$

Magnitude of  $G(j\omega)$ ,

$$|G(j\omega)| = \frac{1}{(u)\sqrt{(u^2 + 4\zeta^2)}} = \frac{1}{\sqrt{(u^4 + 4u^2\zeta^2)}}$$

Phase of  $G(j\omega)$ ,

$$\angle G(j\omega) = -90^\circ - \tan^{-1} \frac{u}{2\zeta}$$

At the gain crossover frequency  $\omega_{gc}$ , the magnitude is unity.

Hence, at  $u=u_{gc}$ ,

$$|G(j\omega_{gc})| = \frac{1}{\sqrt{(u_{gc}^4 + 4u_{gc}^2\zeta^2)}} = 1$$

$$(u_{gc}^4 + 4u_{gc}^2\zeta^2) = 1$$

$$(u_{gc}^4 + 4u_{gc}^2\zeta^2) - 1 = 0$$

Let  $x = u_{gc}^2$

$$x^2 + 4x\zeta^2 - 1 = 0$$

$$x = \frac{-4\zeta^2 \pm \sqrt{16\zeta^4 + 4}}{2} = -2\zeta^2 \pm \sqrt{4\zeta^4 + 1}$$

Let us take only the positive sign,

$$x = -2\zeta^2 + \sqrt{4\zeta^4 + 1}$$

Hence,

$$u_{gc} = [-2\zeta^2 + \sqrt{4\zeta^4 + 1}]^{\frac{1}{2}}$$

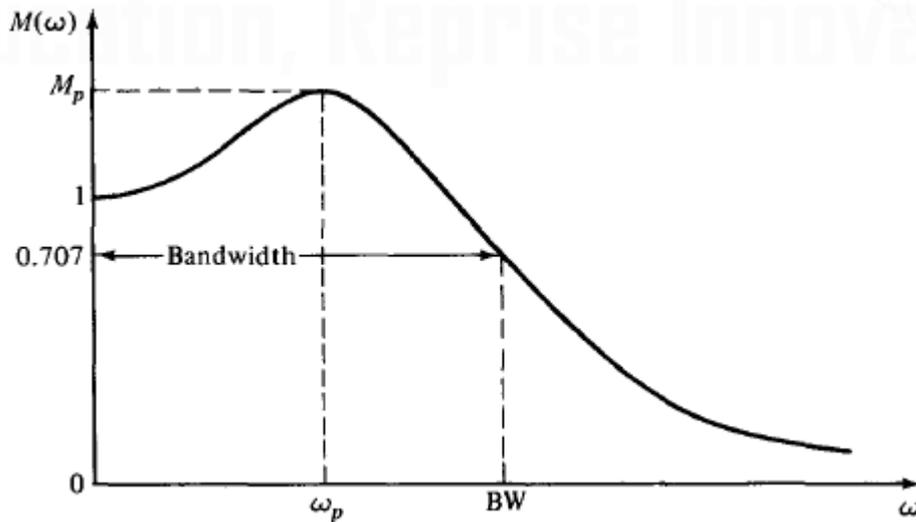
Phase margin,

$$\gamma = 180^\circ + \phi_{gc}$$

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$$\gamma = 180^\circ + \angle G(j\omega_{gc}) = 180^\circ + (-90^\circ - \tan^{-1} \frac{u_{gc}}{2\zeta})$$

$$\gamma = 90^\circ - \tan^{-1} \frac{[-2\zeta^2 + \sqrt{4\zeta^4 + 1}]^{\frac{1}{2}}}{2\zeta}$$



**Figure 3.1.1 Typical magnification curve of a feedback control system**

[Source: "Automatic Control Systems" by Benjamin C. Kuo, Page: 463]

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### 3.3 POLAR PLOT

The polar plot of a sinusoidal transfer function  $G(j\omega)$  is a plot of the magnitude  $|G(j\omega)|$  versus the phase angle of  $G(j\omega)$  on polar coordinates as  $\omega$  is varied from zero to infinity. Thus, the polar plot is the locus of vectors  $|G(j\omega)|$  as  $\omega$  is varied from zero to infinity. The polar plot is also called Nyquist plot. It is a graphical method of determining stability of feedback control systems by using the polar plot of their open-loop transfer functions. Polar plot is a plot to be drawn between magnitude and phase. Polar plot is a plot of magnitude of  $G(j\omega)$  versus the phase of  $G(j\omega)$  in polar co-ordinates. But the magnitudes are presented with normal values only. The Polar plot is a plot, which can be drawn between the magnitude and the phase angle of  $G(j\omega)$   $H(j\omega)$  by differentiating  $g$   $\omega$  from zero to  $\infty$ . The polar graph sheet is described in below mentioned image. This graph sheet includes various concentric circles and radial lines. The concentric circles and the radial lines are considered as the magnitudes and phase angles.

- Angles are highlighted with positive values in anti-clock wise direction.
- Mark angles with negative values in clockwise direction.

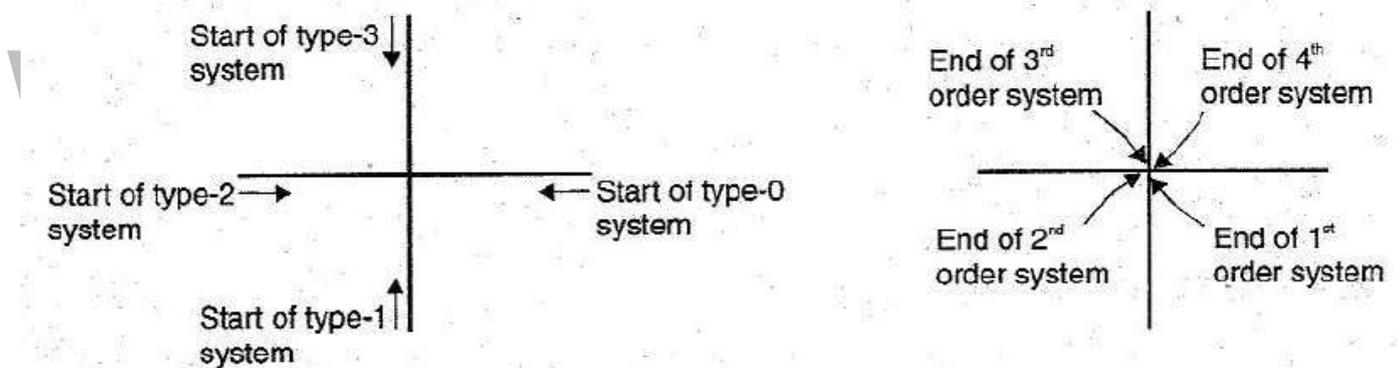
The polar plot is usually plotted on a polar graph sheet. The polar graph sheet has concentric circles and radial lines. The circles represent the magnitude and the radial lines represent the phase angles. Each point on the polar graph has a magnitude and phase angle. The magnitude of a point is given by the value of the circle passing through that point and the phase angle is given by the radial line passing through that point. In polar graph sheet a positive phase angle is measured in anticlockwise from the reference axis ( $0^\circ$ ) and a negative angle is measured clockwise from the reference axis ( $0^\circ$ ). In order to plot the polar plot, magnitude and phase of  $G(j\omega)$  are computed for various values of  $\omega$  and tabulated. Usually the choice of frequencies are corner frequencies and frequencies around corner frequencies. Choose proper scale for the magnitude circles. Fix all the points on polar graph sheet and join the points by smooth curve, write the frequency corresponding to each point of the plot. Alternatively, if  $G(j\omega)$  can be expressed in rectangular coordinates as,

$$G(j\omega) = G_R(j\omega) + jG_I(j\omega)$$

where,  $G_R(j\omega)$  = Real part of  $G(j\omega)$ ,  $G_I(j\omega)$  = Imaginary part of  $G(j\omega)$

Then the polar plot can be plotted in ordinary graph sheet between  $G_R(j\omega)$  and  $G_I(j\omega)$  by varying  $\omega$  from 0 to infinity. In order to plot the polar plot on ordinary graph sheet, the magnitude and phase of  $G(j\omega)$  are computed for various values of  $\omega$ . Then convert the polar coordinates to rectangular coordinates using  $P \rightarrow R$  conversion (polar to rectangular conversion) in the calculator. Sketch the polar plot using rectangular coordinates. For minimum phase transfer function with only poles, type number of the system determines the quadrant at which the polar plot starts and the order of the system determines quadrant at which the polar plot ends. The minimum phase systems are systems with all poles and zeros on left half of s-plane. The start and end of polar plot of all pole minimum phase system are shown in figures respectively. Some typical sketches of polar plot are shown in table. The change in shape of polar plot can be predicted due to addition of a pole or zero.

1. When a pole is added to s system, the polar plot end point will shift by  $-90^\circ$ .
2. When a zero is added to s system, the polar plot end point will shift by  $+90^\circ$ .



**Figure 3.3.1 Start and end of polar plot of all pole minimum phase system**

[Source: "Control Systems" by A Nagoor Kani, Page: 3.38]

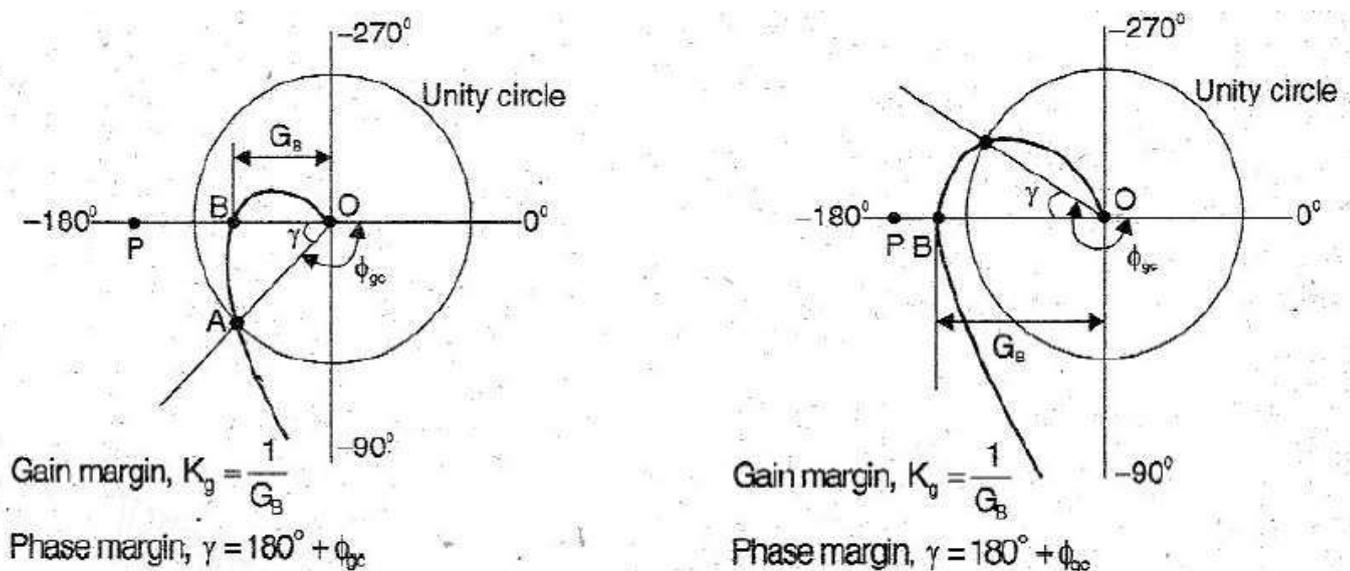
### RULES FOR DRAWING POLAR PLOT

- ✓ Substitute,  $s=j\omega$  in the open loop transfer function.
- ✓ Write the expressions for magnitude and the phase of  $G(j\omega)$   $H(j\omega)$ .
- ✓ Find the starting magnitude and the phase of  $G(j\omega)$   $H(j\omega)$  by substituting  $\omega=0$ . So, the polar plot starts with this magnitude and the phase angle.
- ✓ Find the ending magnitude and the phase of  $G(j\omega)$   $H(j\omega)$  by substituting  $\omega=\infty$ . So, the polar plot ends with this magnitude and the phase angle.
- ✓ Check whether the polar plot intersects the real axis, by making the imaginary term of  $G(j\omega)$   $H(j\omega)$  equal to zero and find the value(s) of  $\omega$ .

- ✓ Check whether the polar plot intersects the imaginary axis, by making real term of  $G(j\omega) H(j\omega)$  equal to zero and find the value(s) of  $\omega$ .
- ✓ For drawing polar plot more clearly, find the magnitude and phase of  $G(j\omega) H(j\omega)$  by considering the other value(s) of  $\omega$ .

### DETERMINATION OF GAIN MARGIN AND PHASE MARGIN FROM POLAR PLOT

The gain margin is defined as the inverse of the magnitude of  $G(j\omega)$  at phase crossover frequency. The phase crossover frequency is the frequency at which the phase of  $G(j\omega)$  is  $180^\circ$ . Let the polar plot cut the  $180^\circ$  axis at point B and the magnitude circle passing through the point B be  $G_B$ . Now the gain margin,  $K_g = 1/ G_B$ . If the point B lies within unity circle, the gain margin is positive otherwise negative. If the polar plot is drawn in ordinary graph sheet using rectangular coordinates then the point B is the cutting point of  $G(j\omega)$  locus with negative real axis and  $K_g = 1/|G_B|$  where  $G_B$  is the magnitude corresponding to point B). The phase margin is defined as, phase margin,  $\gamma = 180^\circ + \Phi_{gc}$  is the phase angle of  $G(j\omega)$  at gain crossover frequency. The gain crossover frequency is the frequency at which the magnitude of  $G(j\omega)$  is unity. Let the polar plot cut the unity circle at point A as shown in figures. Now the phase margin,  $\gamma$  is given by  $\angle AOP$ , i.e.,  $\angle AOP$  is below  $-180^\circ$  axis then the phase margin is positive and if it is above  $-180^\circ$  axis then the phase margin is negative.



**Figure 3.3.2 Polar plot with positive and negative gain and phase margins**

[Source: "Control Systems" by A Nagoor Kani, Page: 3.41]

## GAIN ADJUSTMENT USING POLAR PLOT

### To determine K for specified GM

Draw  $G(j\omega)$  locus with  $K = 1$ . Let it cut the  $-180^\circ$  axis at point B corresponding to gain of  $G_B$ . Let the specified gain margin be  $x$  db. For this gain margin, the  $G(j\omega)$  locus will cut  $-180^\circ$  at point A whose magnitude is  $G_A$ .

$$20 \log \frac{1}{G} = x$$

$$\log \frac{1^A}{G} = \frac{x}{20}$$

$$\frac{G_A}{1} = 10^{\frac{x}{20}}$$

$$\frac{1}{G_A} = 10^{-\frac{x}{20}}$$

$$G_A = \frac{1}{10^{\frac{x}{20}}}$$

Now the value of K is given by,  $K = G_A/G_B$

If,  $K > 1$ , then the system gain should be increased.

If  $K < 1$ , then the system gain should be reduced.

### To determine K for specified PM

Draw  $G(j\omega)$  locus with  $K = 1$ . Let it cut the unity circle at point B. (The gain at point B is  $G_B$  and equal to unity). Let the specified phase margin be  $x^\circ$ . For a phase margin of  $x^\circ$ , let  $\Phi_{gcx}$  be the phase angle of  $G(j\omega)$  at gain crossover frequency.

$$x^\circ = 180^\circ + \phi_{gcx}$$

$$\phi_{gcx} = x^\circ - 180^\circ$$

In the polar plot, the radial line corresponding to  $\phi_{gcx}$  will cut the locus of  $G(j\omega)$  with  $K = 1$  at point A and the magnitude corresponding to that point be  $G_A$ .

$$\text{Now, } K = G_B/G_A = 1/G_A \text{ (since } G_B = 1)$$