2.8   EFFECTS OF P, PI, PID MODES OF FEEDBACK CONTROL

PROPORTIONAL CONTROLLER (P-Controller)

The proportional controller is a device that produces a control signal, u(t) proportional to the input error signal, e(t)

$$u(t) \propto e(t)$$

$$u(t) = K_p e(t)$$

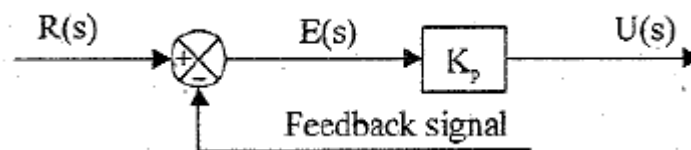where, K_p = Proportional gain or constant

On taking Laplace transform of equation, we get,

$$U(s) = K_p E(s)$$

Transfer function,

$$\frac{U(s)}{E(s)} = K_p$$

The equation gives the output of the P-controller for the input E(s) and it is the transfer function of P-controller. The block diagram of the P-controller is shown in the figure 2.8.1.

**Figure 2.8.1 Block diagram of proportional controller**

[Source: "Control Systems" by Nagoor Kani, Page: 2.79]

From the equation, we can conclude that the proportional controller amplifies the error signal by an amount K_p. Also the introduction of the controller on the system increases the loop gain by an amount K_p. The increase in loop gain improves the steady state tracking accuracy, disturbance signal rejection and the relative stability and also makes the system less sensitive to parameter variations. But increasing the gain to very large values mau lead to instability of the system. The drawback in P-controller is that it leads to a constant steady state error.

Example of Electronic P-controller

The proportion controller can be realized by an amplifier with adjustable gain. Either the non- inverting operational amplifier or the inverting operational amplifier

followed by sign changer will work as a proportional controller. The op-amp proportional controller is shown in the figures 2.8.2.
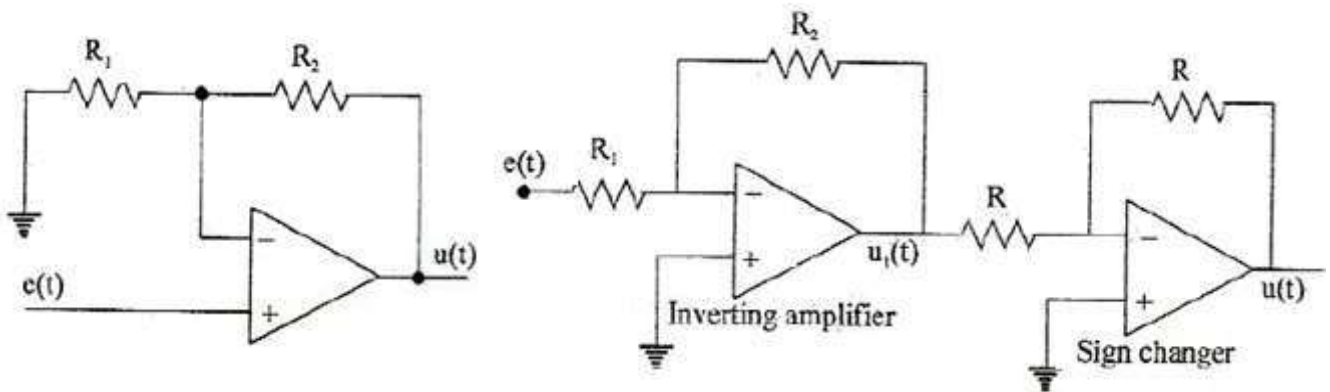

Figure 2.8.2 P-controller using non-inverting and inverting amplifier

[Source: "Control Systems" by Nagoor Kani, Page: 2.80]

By deriving the transfer function of the controller shown in figures and comparing with the transfer function of P-controller defined by equation, it can be shown that they work as P-controllers.

Analysis of P-controller

In figure 2.8.2, the input e(t) is applied to positive input. By symmetry of op-amp the voltage of negative input is also e(t). Also, we assume an ideal op-amp so that input current is zero. Based on the above assumptions the equivalent circuit of the controller is shown in figure 2.8.3.
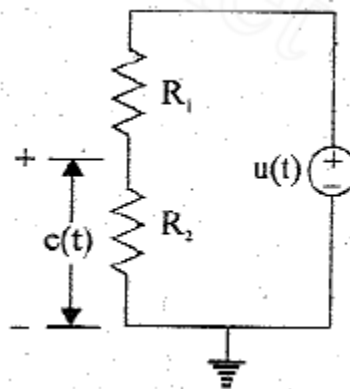
Figure 2.8.3 Equivalent circuit of P-controller

[Source: "Control Systems" by Nagoor Kani, Page: 2.80]

By voltage division rule,

$$e(t) = \frac{R_1}{R_1 + R_2} u(t)$$

On taking Laplace transform of equation we get,

$$\frac{U(s)}{E(s)} = \frac{R_1 + R_2}{R_1}$$

The equation is the transfer function of op-amp P-controller. On comparing, we get,
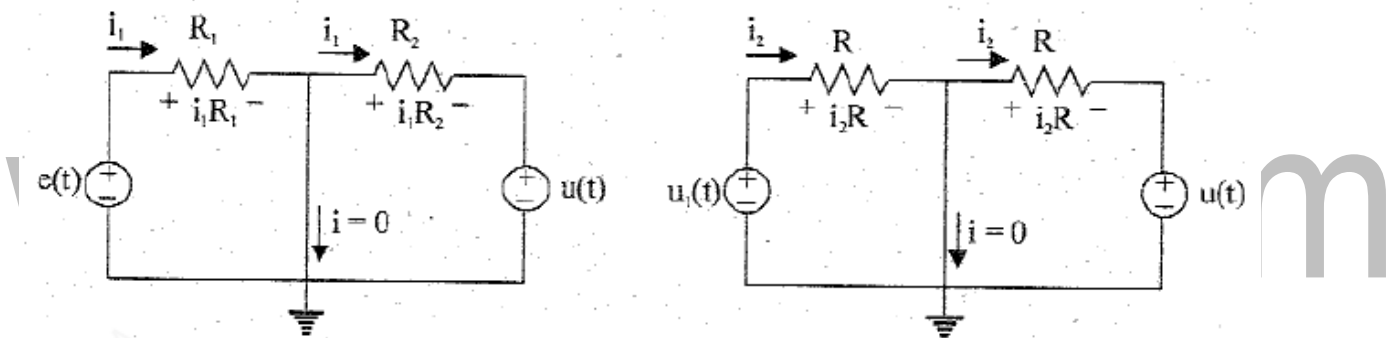
$$K_p = \frac{R_1 + R_2}{R_1}$$

Therefore, by adjusting the values of R_1 and R_2 the value of gain, K_p can be varied.

Analysis of P-controller

The assumption made in op-amp circuit analysis are,

    1. The voltages at both inputs are equal

    2. The input current is zero

Based on the above assumptions, the equivalent circuit of op-amp amplifier and sign changer are shown in figure 2.8.4.

Figure 2.8.4 Equivalent circuit of amplifier and sign changer

[Source: "Control Systems" by Nagoor Kani, Page: 2.81]

From the circuit,

$$e(t) = i_1 R_1$$
$$u_1(t) = -i_1 R_2$$

Substitute for i_1,

$$u_1(t) = - \frac{e(t)}{R_1} R_2$$

Also, from the circuit,

$$u(t) = -i_2 R$$
$$u_1(t) = i_2 R$$

Substitute for i_2,

$$u_1(t) = -u(t)$$

On equating the equations we get,

$$u(t) = \frac{e(t)}{R_1} R_2$$

On taking Laplace transform of equation we get,

$$\frac{U(s)}{E(s)} = \frac{R_2}{R_1}$$

The equation is the transfer function of op-amp P-controller. On the comparing equations, Proportional gain,

$$K_p = \frac{R_2}{R_1}$$

Therefore, by adjusting the values of R_1 and R_2 the value of gain K_p can be varied.

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INTEGRAL CONTROLLER (I-Controller)

The integral controller is a device that produces a control signal u(t) which is proportional to integral of the input error signal, e(t).
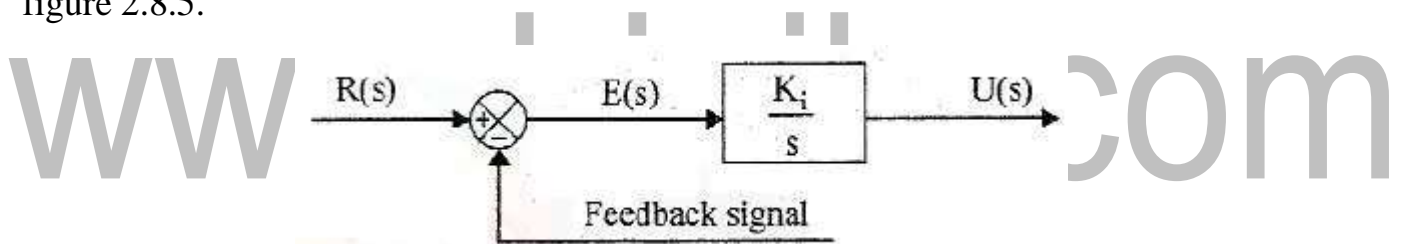
In I-controller

$$u(t) \propto \int e(t)dt$$

$$u(t) = K_i \int e(t)dt$$

where K_i = integral gain or constant

On taking Laplace transform of equation with zero initial conditions we get,

$$\frac{U(s)}{E(s)} = \frac{K_i}{s}$$

The equation gives the output of the I-controller for the input E(s) and equation is the transfer function of the I-controller, the block diagram of I-controller is shown in the figure 2.8.5.


Figure 2.8.5 Block diagram of integral controller

[Source: "Control Systems" by Nagoor Kani, Page: 2.82]

The integral controller removes or reduces the steady error without the need for manual reset. Hence the I-controller is sometimes called automatic reset. The drawback in integral controller is that it may lead to oscillatory response of increasing or decreasing amplitude which is undesirable and the system may become unstable.

Example of electronic I-controller

The integral controller can be realized by an integrator using op-amp followed by a sign changer as shown in figure 2.8.6.
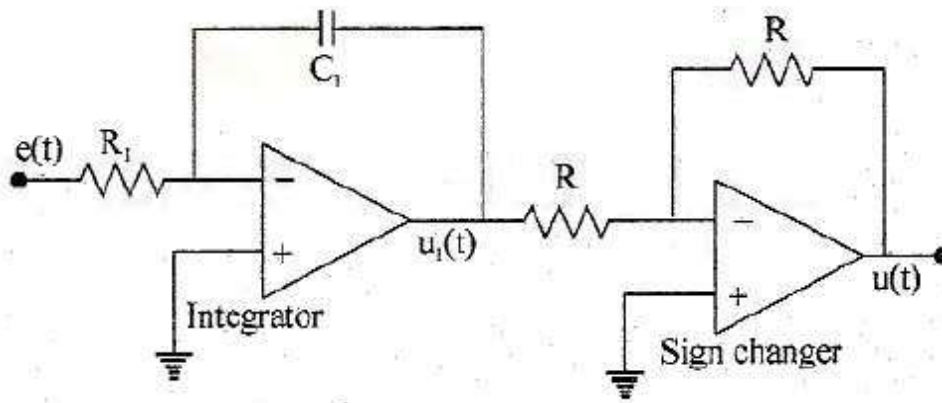
Figure 2.8.6 I-controller using inverting amplifier

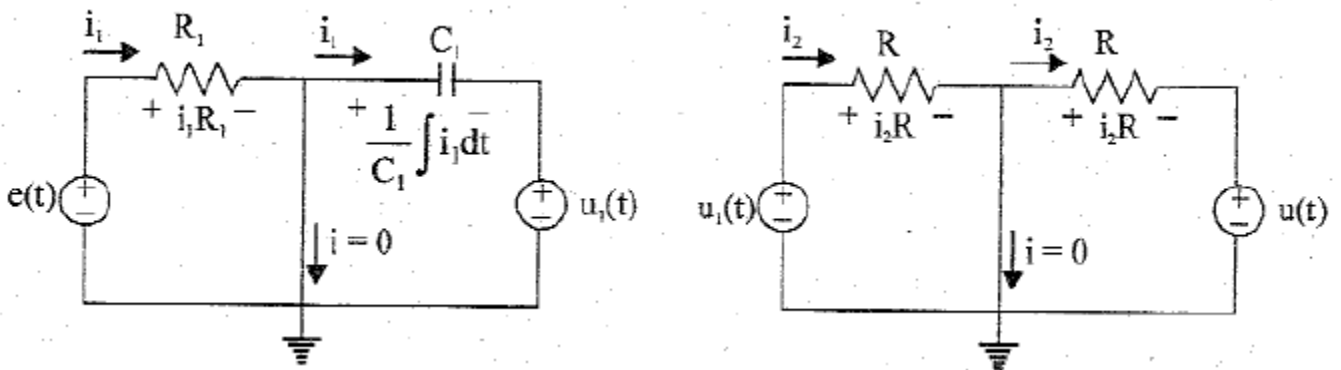*[Source: "Control Systems" by Nagoor Kani, Page: 2.82]*

By deriving the transfer function of the controller shown in figure and comparing with the transfer function of I-controller defined by equation.

Analysis of I-controller

The assumptions made in op-amp circuit analysis are,

1. The voltages of both inputs are equal
2. The input current is zero.

Based on the above assumptions, the equivalent circuit of op-amp amplifier and sign changer are shown in figure 2.8.7.

Figure 2.8.7 Equivalent circuit of amplifier and sign changer

[Source: "Control Systems" by Nagoor Kani, Page: 2.83]

From the circuit,

$$e(t) = i_1 R_1$$

$$u_1(t) = -\frac{1}{C_1}\int i_1 \, dt$$

Substitute for i_1,

$$u_1(t) = -\frac{1}{R_1 C_1}\int e(t)dt$$

Also, from the circuit,

$$u(t) = -i_2 R$$

$$u_1(t) = i_2 R$$

Substitute for i_2,

$$u_1(t) = -u(t)$$

On equating equations we get

$$u(t) = \frac{1}{R_1 C_1}\int e(t)dt$$

On taking Laplace transform of equation we get,

$$\frac{U(s)}{E(s)} = \frac{1}{sR_1 C_1}$$

The equation is the transfer function of op-amp P-controller. On the comparing equations, Integral gain,

$$K_i = \frac{1}{R_1 C_1}$$

Therefore, by adjusting the values of R_1 and C_1 the value of gain K_i can be varied.

PROPORTIONAL PLUS INTEGRAL CONTROLLER (PI-CONTROLLER)

The proportional plus integral controller (PI controller) produces and output signal consisting of two terms: *one proportional to error signal and the other proportional to the integral of error signal.*
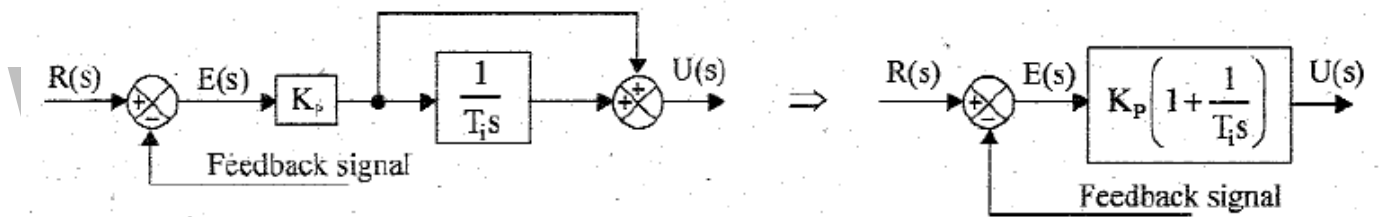
In PI controller,

$$u(t) \propto [e(t) + \int e(t)dt]$$

$$u(t) = K_p e(t) + \frac{K_p}{T_i} \int e(t)dt$$

On taking Laplace transform of equation with zero initial conditions, we get,

$$\frac{U(s)}{E(s)} = K_p \left(1 + \frac{1}{T_i s}\right)$$

The equation gives the output of the PI-controller for the input E(s) and it is the transfer function of PI-controller. The block diagram of PI-controller is shown in figure 2.8.8.

**Figure 2.8.8 Block diagram of PI controller**

[Source: "Control Systems" by Nagoor Kani, Page: 2.84]

The advantages of both P-controller and I –controller is combined in PI-controller. The proportional action increases the loop gain and makes the system less sensitive to variations of system parameters. The integral action eliminates or reduces the steady state error. The integral control action is adjusted by varying the integral time. The change in value of K_p affects both the proportional and integral parts of control action. The inverse of the integral time T_i is called the reset rate.

Example of Electronic PI-controller

The PI controller can be realized by an op-amp differentiator with gain followed by a sign changer as shown in figure 2.8.9.
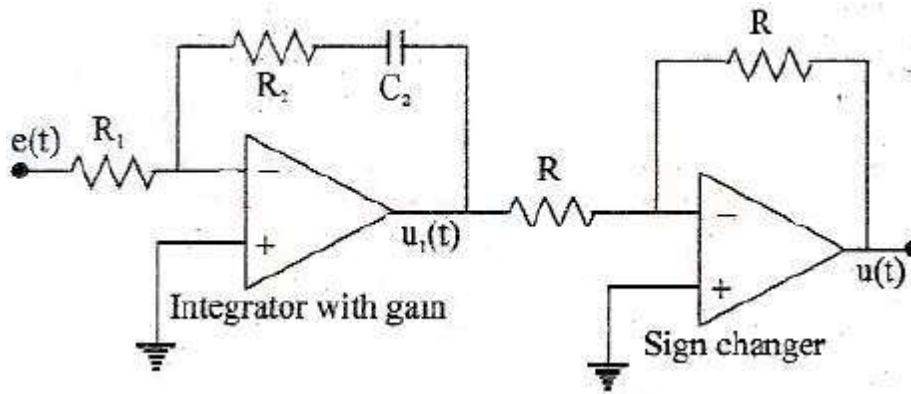
Figure 2.8.9 PI-controller using inverting amplifier

[Source: "Control Systems" by Nagoor Kani, Page: 2.84]

By deriving the transfer function of the controller shown in figure and comparing with the transfer function of PI-controller defined by equation, it can be proved that the circuit shown in figure will work as PI-controller.

Analysis of PI-controller

The assumptions made in op-amp circuit analysis are,

    1. The voltages of both inputs are equal

    2. The input current is zero.

Based on the above assumptions, the equivalent circuit of op-amp amplifier and sign changer are shown in figure 2.8.10.
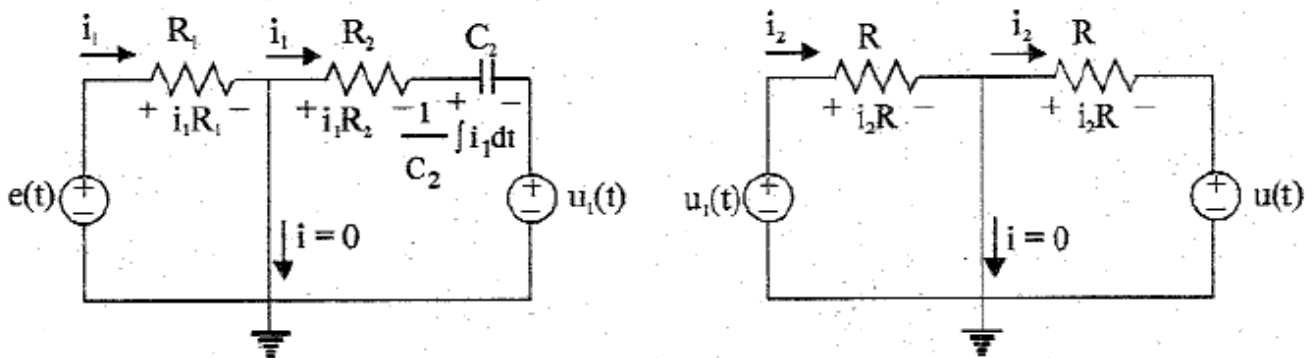
Figure 2.8.10 Equivalent circuit of amplifier and sign changer

[Source: "Control Systems" by Nagoor Kani, Page: 2.85]

From the circuit,

$$e(t) = i_1 R_1$$

$$u_1(t) = -i_1 R_2 - \frac{1}{C_2}\int i_1\, dt$$

Substitute for i_1,

$$u_1(t) = -\frac{e(t)}{R_1}R_2 - \frac{1}{R_1 C_2}\int e(t)dt$$

Also, from the circuit,

$$u(t) = -i_2 R$$

$$u_1(t) = i_2 R$$

Substitute for i_2,

$$u_1(t) = -u(t)$$

On equating equations we get

$$u(t) = \frac{e(t)}{R_1}R_2 + \frac{1}{R_1 C_2}\int e(t)dt$$

On taking Laplace transform of equation we get,

$$\frac{U(s)}{E(s)} = \frac{R_2}{R_1}\left(1 + \frac{1}{sR_2 C_2}\right)$$

The equation is the transfer function of op-amp P-controller. On the comparing equations,

Proportional gain,

$$K_p = \frac{R_2}{R_1}$$

Integral time,

$$T_i = R_2 C_2$$

By varying the values of R_1 and R_2, the value of gain K_p and T_i can be adjusted.

PROPORTIONAL PLUS DERIVATIVE CONTROLLER (PD-CONTROLLER)

The PD controller produces and output signal consisting of two terms: *one proportional to error signal, the other one proportional to derivatives of error signal.*
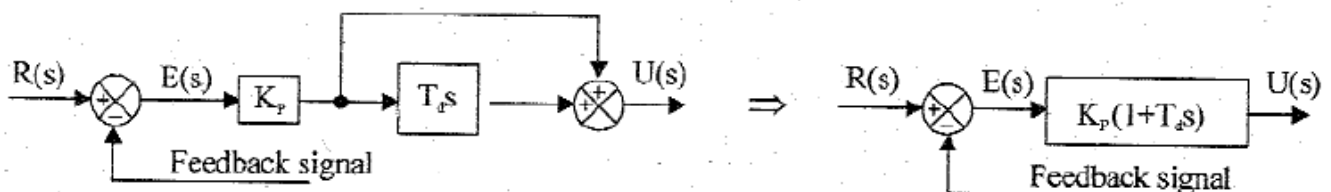
In PD controller,

$$u(t) \propto [e(t) + \frac{d}{dt} e(t)]$$

$$u(t) = K_p e(t) + K_p T_d \frac{d}{dt} e(t)$$

On taking Laplace transform of equation with zero initial conditions, we get,

$$\frac{U(s)}{E(s)} = K_p(1 + T_d s)$$

The equation gives the output of the PD-controller for the input E(s) and it is the transfer function of PD-controller. The block diagram of PD-controller is shown in figure 2.8.11.


Figure 2.8.11 Block diagram of PD controller

[Source: "Control Systems" by Nagoor Kani, Page: 2.86]

The derivative control acts on a rate of change of error and not on the actual error signal. The derivative control action is effective only during transient periods and so it does not produce corrective measures for any constant error. Hence the derivative controller is never used alone, but it is employed in association with proportional and integral controllers. The derivative controller does not affect the steady-state error directly but anticipates the error, initiates an early corrective action and tends to increase the stability of the system. While derivative control action has an advantage of being anticipatory it has the disadvantage that it amplifies noise signals and may cause a saturation effect in the actuator. The derivative control action is adjusting by varying the derivative time. The change in the value of K_p affects both the proportional and derivative parts of control action. The derivative control is also called rate control.

Example of Electronic PD-controller

The PD controller can be realized by an op-amp amplifier with integral and derivative action followed by a sign changer as shown in figure 2.8.12.
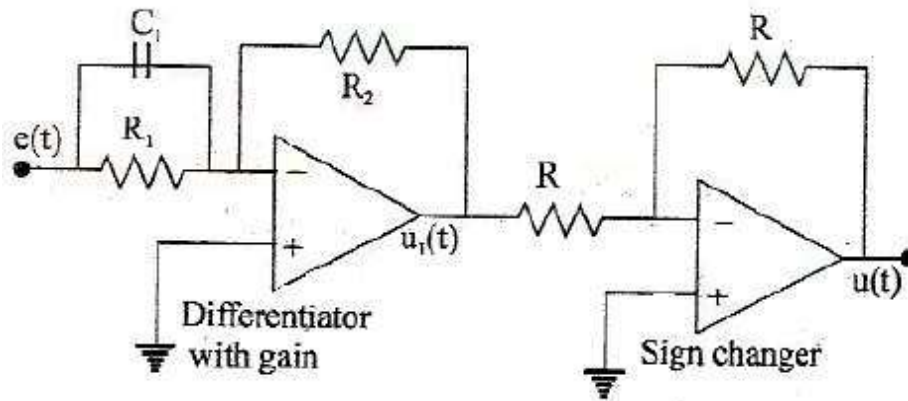
Figure 2.8.12 PD-controller using inverting amplifier

[Source: "Control Systems" by Nagoor Kani, Page: 2.86]

By deriving the transfer function of the controller shown in figure and comparing with the transfer function of PD-controller defined by equation, it can be proved that the circuit shown in figure will work as PD-controller.

Analysis of PD-controller

The assumptions made in op-amp circuit analysis are,

1. The voltages of both inputs are equal
2. The input current is zero.

Based on the above assumptions, the equivalent circuit of op-amp amplifier and sign changer are shown in figure 2.8.13.
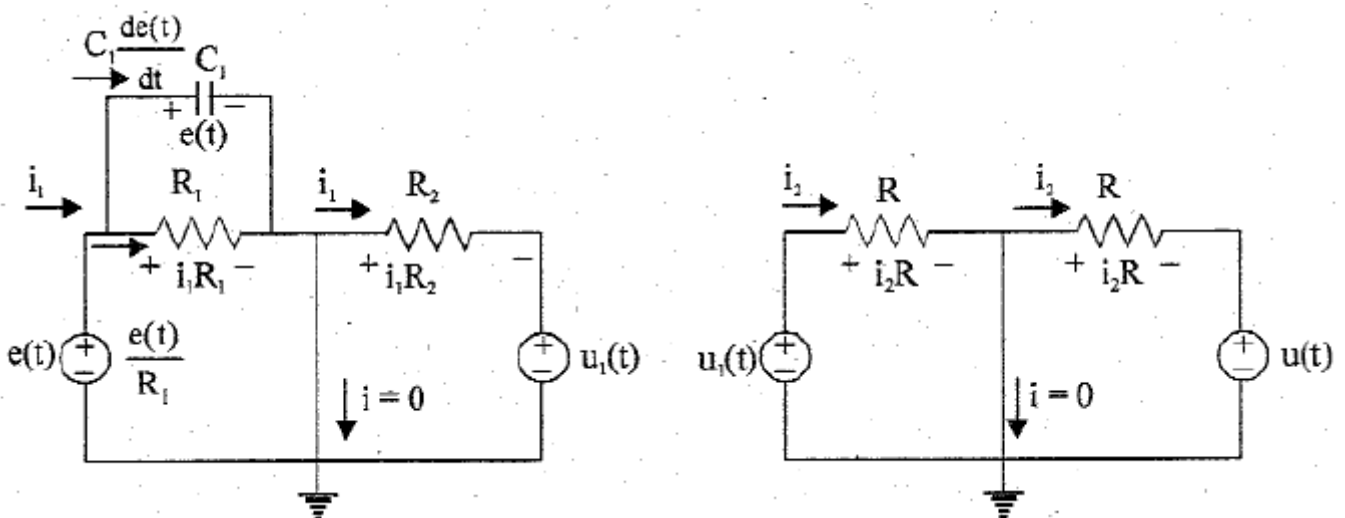
Figure 2.8.13 Equivalent circuit of amplifier and sign changer

[Source: "Control Systems" by Nagoor Kani, Page: 2.87]

From the circuit,

$$i_1 = \frac{e(t)}{R_1} + C_1 \frac{de(t)}{dt}$$

$$u_1(t) = -i_1 R_2$$

Substitute for i_1,

$$u_1(t) = -\frac{e(t)}{R_1} R_2 - R_2 C_1 \frac{d}{dt} e(t)$$

Also, from the circuit,

$$u(t) = -i_2 R$$
$$u_1(t) = i_2 R$$

Substitute for i_2,

$$u_1(t) = -u(t)$$

On equating the equations, we get,

$$u(t) = \frac{e(t)}{R_1} R_2 + R_2 C_1 \frac{d}{dt} e(t)$$

On taking Laplace transform of equation we get,

$$\frac{U(s)}{E(s)} = \frac{R_2}{R_1}(1 + sR_1 C_1)$$

The equation is the transfer function of op-amp P-controller. On the comparing equations,

Proportional gain,

$$K_p = \frac{R_2}{R_1}$$

Derivative time,

$$T_d = R_1 C_1$$

By varying the values of R_1 and R_2, the value of K_p and T_d are adjusted.

PROPORTIONAL PLUS INTEGRAL PLUS DERIVATIVE (PID) CONTROLLER

The PID controller produces and output signal consisting of two terms: *one proportional to error signal, another one proportional to the integral of error signal and the third one proportional to derivatives of error signal.*
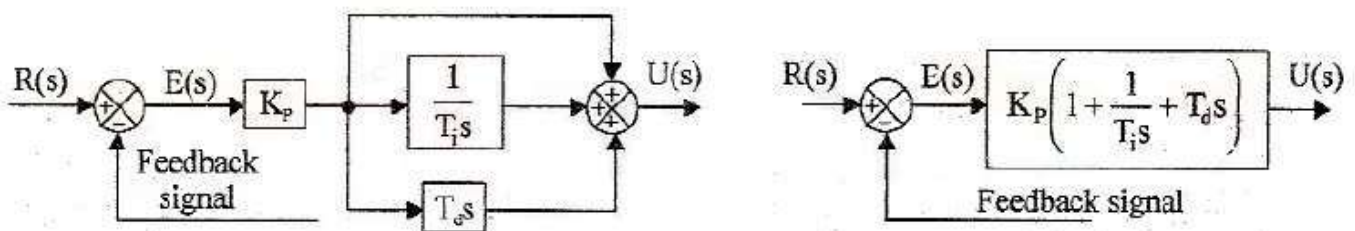
$$u(t) \propto [e(t) + \int e(t)dt + \frac{d}{dt}e(t)]$$

$$u(t) = K_p\, e(t) + \frac{K_p}{T_i}\int e(t)dt + K_p T_d \frac{d}{dt}e(t)$$

On taking Laplace transform of equation with zero initial conditions, we get,

$$\frac{U(s)}{E(s)} = K_p\left(1 + \frac{1}{T_i s} + T_d s\right)$$

The equation gives the output of the PID-controller for the input E(s) and it is the transfer function of PID-controller. The block diagram of PID-controller is shown in figure 2.8.14.

Figure 2.8.14 Block diagram of PID controller

[Source: "Control Systems" by Nagoor Kani, Page: 2.88]

The combination of proportional control action, integral control action and derivative control action is called PID-control action. This combined action has the advantages of the each of the three individual control actions. The proportional controller stabilizes the gain but produces a steady state error. The integral controller reduces or eliminates the steady state error. The derivative controller reduces the rate of change of error.

Example of Electronic PID-controller

The PID controller can be realized by an op-amp amplifier with integral and derivative action followed by a sign changer as shown in figure 2.8.15.
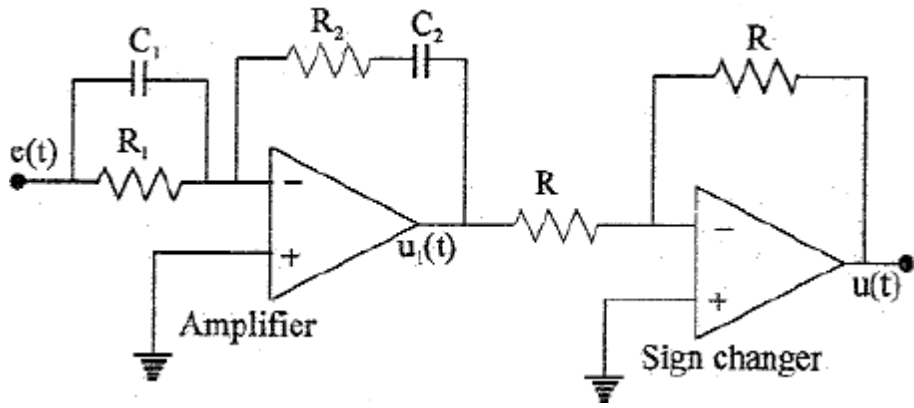
Figure 2.8.15 PID-controller using inverting amplifier

[Source: "Control Systems" by Nagoor Kani, Page: 2.88]

By deriving the transfer function of the controller shown in figure and comparing with the transfer function of PID-controller defined by equation, it can be proved that the circuit shown in figure will work as PID-controller.

Analysis of PID-controller

The assumptions made in op-amp circuit analysis are,

      3. The voltages of both inputs are equal

      4. The input current is zero.

Based on the above assumptions, the equivalent circuit of op-amp amplifier and sign changer are shown in figure 2.8.16.
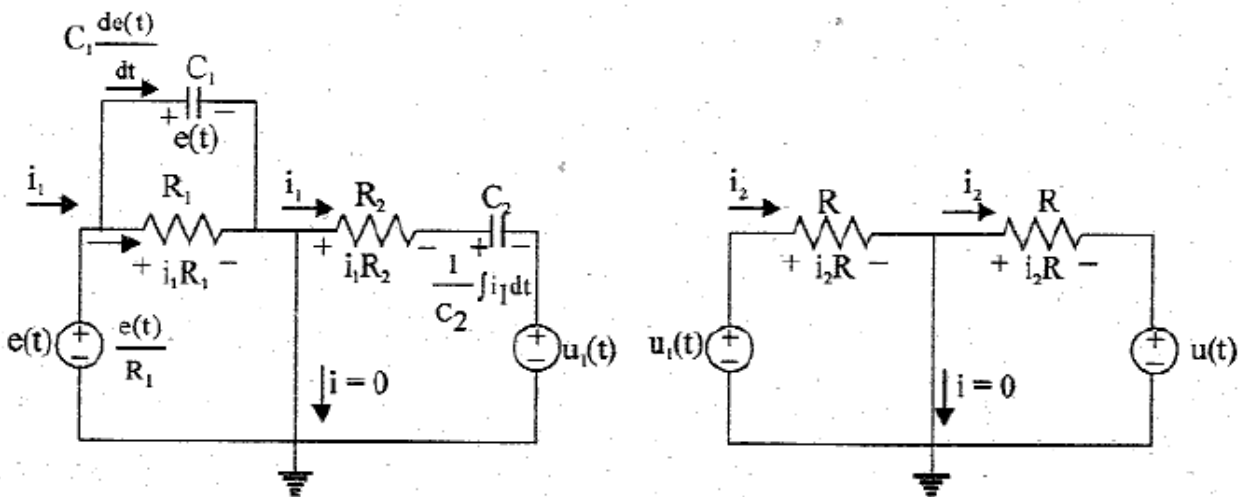
Figure 2.8.16 Equivalent circuit of amplifier and sign changer

[Source: "Control Systems" by Nagoor Kani, Page: 2.89]

From the circuit,

$i_1 =$

On taking Laplace transform of equation with zero initial conditions, we get,

$$I_1(s) = \left(\frac{1}{R_1} + C_1 s\right) E(s)$$

Also, from the circuit,

$$u_1(t) = -i_1 R_2 - \frac{1}{2} \int i_1 \, dt$$

On taking Laplace transform of equation with zero initial conditions, we get,

$$U_1(s) = -I_1(s)R_2 - \frac{1}{sC_2}I_1(s)$$

Substitute for i_1, from equations

$$U_1(s) = -\left(\frac{R_2}{R_1} + \frac{C_1}{C_2} + \frac{1}{R_1 C_2 s} + R_2 C_1 s\right) E(s)$$

Also, from the circuit,

$$u(t) = -i_2 R$$
$$u_1(t) = i_2 R$$

Substitute for i_2,

$$u_1(t) = -u(t)$$

On equating the equations, we get,

$$\frac{U(s)}{E(s)} = \frac{R_2}{R_1}\left(1 + \frac{R_1 C_1 + R_2 C_2}{R_2 C_2} + \frac{1}{R_2 C_2 s} + R_1 C_1 s\right)$$

The equation is the transfer function of op-amp PID-controller. On the comparing, we get,

Proportional gain, $K_p = \dfrac{R_2}{R_1}$

Derivative time, $T_d = R_1 C_1$

Integral time, $T_i = R_2 C_2$

Also, $\dfrac{R_1 C_1 + R_2 C_2}{} = 1$

$R_2 C_2$

By varying the values of R_1 and R_2, the value of K_p, T_d and T_i are adjusted.

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2.5 ERROR COEFFICIENTS

There are two different types of error coefficient representation namely,

a) Static error constants

b) Generalized error coefficients

STATIC ERROR CONSTANTS

$$Positional\ error\ constant,\ K_p = \lim_{s \to 0} G(s)H(s)$$

$$Velocity\ error\ constant,\ K_v = \lim_{s \to 0} sG(s)H(s)$$

$$Acceleration\ error\ constant,\ K_a = \lim_{s \to 0} s^2 G(s)H(s)$$

GENERALIZED ERROR COEFFICIENTS

$$C_o = \lim_{s \to 0} F(s)$$

$$C_1 = \lim_{s \to 0} \frac{dF(s)}{ds}$$

$$C_2 = \lim_{s \to 0} \frac{d^2 F(s)}{ds^2}$$

where, $F(s) = \dfrac{1}{1+G(s)H(s)}$

Relation between static error constants and generalized error coefficients

$$C_o = \frac{1}{1 + K_p}$$

$$C_1 = \frac{1}{K_v}$$

$$C_2 = \frac{1}{K_a}$$

2.4   FIRST AND SECOND ORDER SYSTEM RESPONSE

Transfer Function

- It is the ratio of Laplace transform of output to Laplace transform of input with zero initial conditions.

- One of the types of modeling a system

- Using first principle, differential equation is obtained

- Laplace Transform is applied to the equation assuming zero initial conditions

Order of a system

- ✓ Order of a system is given by the order of the differential equation governing the system

- ✓ Alternatively, order can be obtained from the transfer function

- ✓ In the transfer function, the maximum power of s in the denominator polynomial gives the order of the system

Dynamic Order of Systems

- Order of the system is the order of the differential equation that governs the dynamic behaviour

- Working interpretation: Number of the dynamic elements / capacitances or holdup elements between a manipulated variable and a controlled variable

- Higher order system responses are usually very difficult to resolve from one another

- The response generally becomes sluggish as the order increases

SYSTEM RESPONSE

First-order system time response

- Transient
- Steady-state

Second-order system time response

- Transient
- Steady-state

FIRST ORDER SYSTEM

Response of First Order System for Unit Step Input

The standard form of closed loop transfer function of first order system is

$$\frac{C(s)}{R(s)} = \frac{1}{1+sT}$$

If the input is unit step, then r(t) and R(s)=1/s

$$C(s) = R(s)\ \frac{1}{1+sT} = \frac{1}{s} \times \frac{1}{1+sT}$$

Applying partial fraction expansion,

$$C(s) = \frac{A}{s} + \frac{B}{1+sT}$$
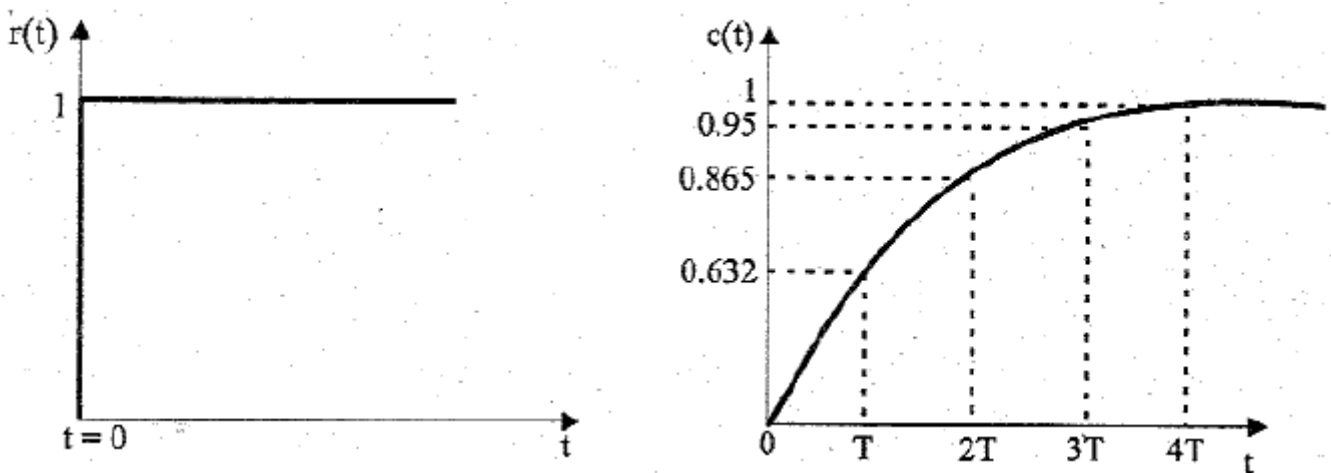
On solving,

$$C(s) = \frac{1}{s} - \frac{1}{s+\frac{1}{T}}$$

On taking inverse Laplace transform, the response in time domain is obtained as,

$$c(t) = 1 - e^{-\frac{t}{T}}$$

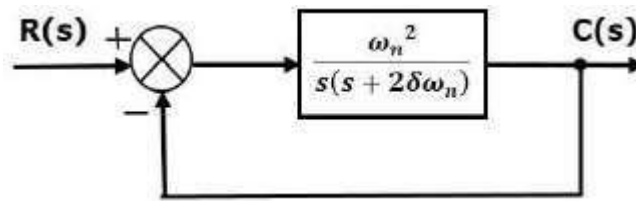Hence, the input and output signal of the first order system is given by,

Figure 2.4.1 Response of first order system to unit step input

[Source: "Control Systems" by Nagoor Kani, Page: 2.20]

SECOND ORDER SYSTEM

LTI second-order system

Figure 2.4.2 Closed loop for second order system

[Source: "Control Systems" by Nagoor Kani, Page: 2.20]

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

$$\frac{C(s)}{R(s)} = \frac{\left(\dfrac{\omega_n^2}{s(s + 2\zeta\omega_n}\right)}{1+\left(\dfrac{\omega_n^2}{s(s+2\zeta\omega_n)}\right)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n\ s + \omega_n^2}$$

where, ζ is the damping ratio, ω_n is the natural frequency

DAMPING RATIO

It is the ratio of critical damping to actual damping.

CHARACTERISTIC EQUATION

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

$$s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

The roots of characteristic equation are:

☐ The two roots are imaginary when $\zeta = 0$ (undamped system)

☐ The two roots are real and equal when $\zeta = 1$ (critically damped system)

☐ The two roots are real but not equal when $\zeta > 1$ (overdamped system)

☐ The two roots are complex conjugate when $0 < \zeta < 1$ (underdamped system)

Response of Second Order System for Unit Step Input

Consider the unit step signal as an input to the second order system. Laplace transform of the unit step signal is

$$R(s) = 1/s$$

Transfer function of the second order closed loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Case 1: Undamped system

When $\zeta = 0$,

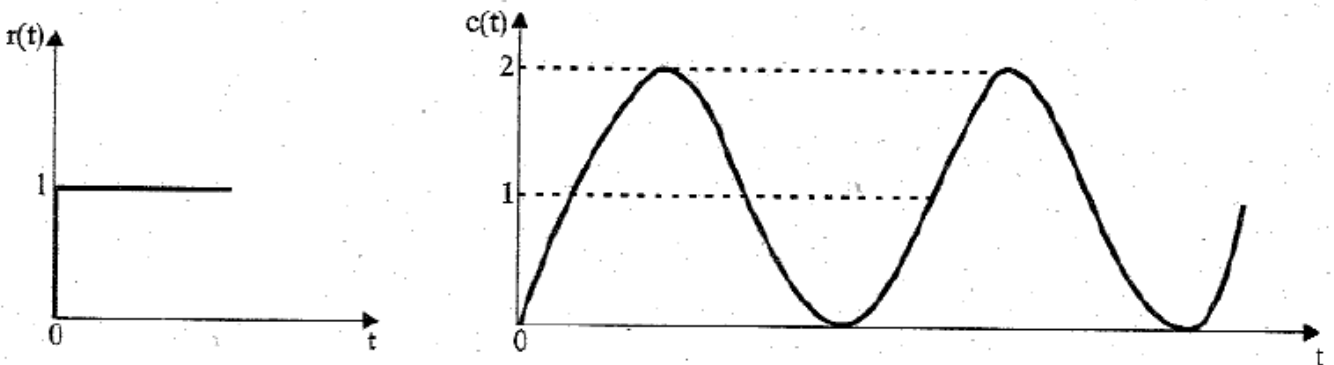$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + \omega_n^2}$$

For unit step input, $R(s) = 1/s$,

$$C(s) = \frac{\omega_n^2}{s^2 + \omega_n^2}\left(\frac{1}{s}\right) = \frac{\omega_n^2}{s(s^2 + \omega_n^2)}$$

Taking inverse Laplace transform,

$$c(t) = 1 - \cos \omega_n t$$

Figure 2.4.3 Response of undamped second order system to unit step input

[Source: "Control Systems" by Nagoor Kani, Page: 2.22]

Case 2: Underdamped system

When $0 < \zeta < 1$,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$s^2 + 2\zeta\omega_n s + \omega^2 = \{s^2 + 2\zeta\omega_n s + (\omega_n)^2\} + \omega_n^2 - (\zeta\omega_n)^2$$

$$= (s + \zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2)$$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{(s + \zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2)}$$

For unit step input, $R(s) = 1/s$,

$$C(s) = \frac{\omega_n^2}{s((s + \zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2))}$$

By applying partial fraction,

$$C(s) = \frac{A}{s} + \frac{Bs + C}{((s + \zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2))}$$

On solving, we get,

$$C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{((s + \zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2))}$$

$$C(s) = \frac{1}{s} - \frac{s + \zeta\omega_n}{((s + \zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2))} - \frac{\zeta\omega_n}{((s + \zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2))}$$

$$C(s) = \frac{1}{s} - \frac{s + \zeta\omega_n}{((s + \zeta\omega_n)^2 + (\omega_n\sqrt{1 - \zeta^2})^2)} - \frac{\zeta\omega_n}{((s + \zeta\omega_n)^2 + (\omega_n\sqrt{1 - \zeta^2})^2)}$$

$$C(s) = \frac{1}{s} - \frac{s + \zeta\omega_n}{((s + \zeta\omega_n)^2 + (\omega_n\sqrt{1 - \zeta^2})^2)}$$

$$\frac{\zeta}{} \qquad \overline{} - \qquad \sqrt{1 - \zeta^2}$$

$$\omega_n\sqrt{1-\zeta^2}$$

$$((s+\zeta\omega_n)^2 + (\omega_n\sqrt{1-\zeta^2})^{\overline{}})^2$$

On taking inverse Laplace transform,

$$c(t) = (1 - e^{-\zeta\omega_n t}\cos\omega_d t - \frac{\zeta}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\omega_d t)$$

$$c(t) = (1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}((\sqrt{1-\zeta^2})\cos\omega_d t + \zeta\sin\omega_d t))$$

We know, $\sin\theta = \sqrt{1-\zeta^2}$, $\cos\theta = \zeta$

$$c(t) = (1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}(\sin\theta\cos\omega_d t + \cos\theta\sin\omega_d t))$$

$$c(t) = (1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}(\sin(\omega_d t + \theta)))$$
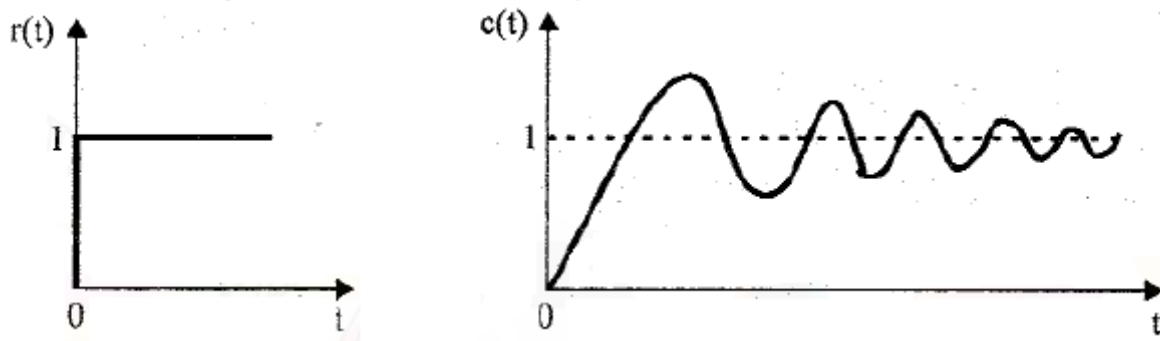
Figure 2.4.4 Response of underdamped second order system to unit step input

[Source: "Control Systems" by Nagoor Kani, Page: 2.24]

Case 3: Critically damped system

When $\zeta = 1$,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\omega_n s + \omega_n^2}$$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{(s + \omega_n)^2}$$

For a step input, R(s)=1/s

$$C(s) = \frac{\omega_n^2}{s(s + \omega_n)^2}$$

By applying partial fractions,

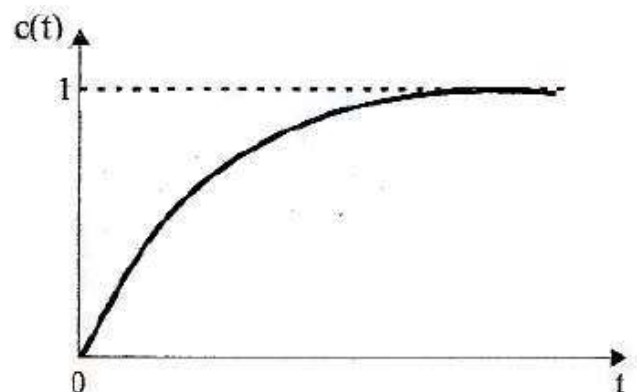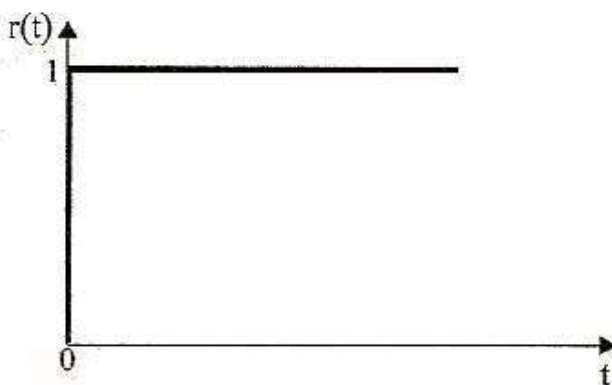$$C(s) = \frac{A}{s} + \frac{B}{s + \omega_n} + \frac{C}{(s + \omega_n)^2}$$

On solving, we get

$$C(s) = \frac{1}{s} - \frac{1}{s + \omega_n} - \frac{\omega_n}{(s + \omega_n)^2}$$

By taking inverse Laplace transform,

$$c(t) = 1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t}$$

Figure 2.4.5 Response of critically damped second order system to unit step input

[Source: "Control Systems" by Nagoor Kani, Page: 2.25]

Case 4: Overdamped system

When $\zeta > 1$,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = \{s^2 + 2\zeta\omega_n s + \omega_n^2 + \zeta^2\omega_n^2 - \zeta^2\omega_n^2\}$$

$$= (s + \zeta\omega_n)^2 - \omega_n^2(\zeta^2 - 1)$$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{(s + \zeta\omega_n)^2 - \omega_n^2(\zeta^2 - 1)}$$

For unit step input, R(s)=1/s,

$$C(s) = \frac{\omega_n^2}{s[(s + \zeta\omega_n)^2 - \omega_n^2(\zeta^2 - 1)]}$$

$$C(s) = \frac{\omega_n^2}{s(s + \zeta\omega_n + \omega_n\sqrt{1 - \zeta^2})(s + \zeta\omega_n - \omega_n\sqrt{1 - \zeta^2})}$$

By applying partial fraction,

$$C(s) = \frac{A}{s} + \frac{B}{(s + \zeta\omega_n + \omega_n\sqrt{1 - \zeta^2})} + \frac{C}{(s + \zeta\omega_n - \omega_n\sqrt{1 - \zeta^2})}$$

By applying inverse Laplace transform,

$$c(t) = [1 + (\frac{1}{2(\zeta + \sqrt{\zeta^2 - 1})(\sqrt{\zeta^2 - 1})})e^{-(\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1})t}$$

$$- (\frac{1}{2(\zeta - \sqrt{\zeta^2 - 1})(\sqrt{\zeta^2 - 1})})e^{-(\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1})t}]$$
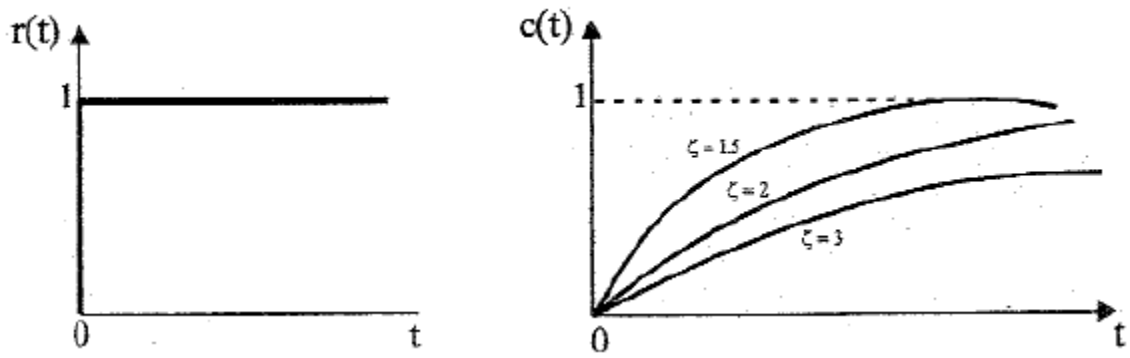
Figure 2.4.6 Response of over damped second order system to unit step input

[Source: "Control Systems" by Nagoor Kani, Page: 2.27]

2.7   ROOT LOCUS CONSTRUCTION

The root locus is a graphical representation in s-domain and it is symmetrical about the real axis. Because the open loop poles and zeros exist in the s-domain having the values either as real or as complex conjugate pairs.

Rules for Construction of Root Locus

The following rule structure is followed for constructing a root locus.

Rule 1 − Locate the open loop poles and zeros in the 's' plane.

Rule 2 − Find the number of root locus branches.

We know that the root locus branches start at the open loop poles and end at open loop zeros. So, the number of root locus branches **N** is equal to the number of finite open loop poles **P** or the number of finite open loop zeros **Z**, whichever is greater. Mathematically, we can write the number of root locus branches **N** as

$$N = P \quad \text{if} \quad P \geq Z$$

$$N = Z \quad \text{if} \quad P < Z$$

Rule 3 − Identify and draw the **real axis root locus branches**.

If the angle of the open loop transfer function at a point is an odd multiple of 180^0, then that point is on the root locus. If odd number of the open loop poles and zeros exist to the left side of a point on the real axis, then that point is on the root locus branch. Therefore, the branch of points which satisfies this condition is the real axis of the root locus branch.

Rule 4 − Find the centroid and the angle of asymptotes.

- If *P=Z*

then all the root locus branches start at finite open loop poles and end at finite open loop zeros.

- If *P>Z*

then *Z* number of root locus branches start at finite open loop poles and end at finite open loop zeros and *P−Z*

number of root locus branches start at finite open loop poles and end at infinite open loop zeros.

- If *P<Z*

then P number of root locus branches start at finite open loop poles and end at finite open loop zeros and *Z−P* number of root locus branches start at infinite open loop poles and end at finite open loop zeros.

So, some of the root locus branches approach infinity, when *P≠Z*. Asymptotes give the direction of these root locus branches. The intersection point of asymptotes on the real axis is known as centroid.

We can calculate the **centroid α** by using this formula,

$$\alpha = \frac{\sum Real\ part\ of\ finite\ open\ loop\ poles - \sum Real\ part\ of\ finite\ open\ loop\ zeros}{P-Z}$$

Angle of asymptotes,

$$\theta = \frac{(2q+1)180^0}{P-Z}$$

where,

$$q=0,1,2,...,(P-Z)-1$$

Rule 5 − Find the intersection points of root locus branches with an imaginary axis.

We can calculate the point at which the root locus branch intersects the imaginary axis and the value of **K** at that point by using the Routh array method and special **case (ii)**.

- If all elements of any row of the Routh array are zero, then the root locus branch intersects the imaginary axis and vice-versa.

- Identify the row in such a way that if we make the first element as zero, then the elements of the entire row are zero. Find the value of **K** for this combination.

- Substitute this **K** value in the auxiliary equation. You will get the intersection point of the root locus branch with an imaginary axis.

Rule 6 − Find Break-away and Break-in points.

- If there exists a real axis root locus branch between two open loop poles, then there will be a break-away point in between these two open loop poles.

- If there exists a real axis root locus branch between two open loop zeros, then there will be a break-in point in between these two open loop zeros.

[Note − Break-away and break-in points exist only on the real axis root locus branches.]

Follow these steps to find break-away and break-in points.

1. Write *K* in terms of *s* from the characteristic equation 1+*G*(*s*)*H*(*s*)=0

2. Differentiate *K* with respect to s and make it equal to zero. Substitute these values of *s* in the above equation.

3. The values of *s* for which the *K* value is positive are the break points.

Rule 7 − Find the angle of departure and the angle of arrival.

The Angle of departure and the angle of arrival can be calculated at complex conjugate open loop poles and complex conjugate open loop zeros respectively.

Angle of departure,

$$\phi_d = 180^0 - \phi$$

Angle of arrival,

$$\phi_a = 180^0 + \phi$$

where,

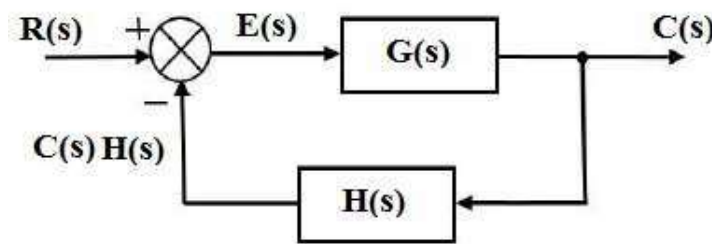$$\phi = \sum \phi_P - \sum \phi_Z$$

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2.6   STEADY STATE ERROR

The deviation of the output of control system from desired response during steady state is known as steady state error. It is represented as e_{ss}. We can find steady state error using the final value theorem as follows.

$$e_{ss}=\lim_{t\to\infty} e(t)$$

$$e_{ss}=\lim_{s\to0} sE(s)$$

where, E(s) is the Laplace transform of the error signal, e(t)

Figure 2.6.1 Closed loop control system

[Source: "Control Systems Engineering" by I J Nagrath, M Gopal, Page: 213]

$$C(s) = G(s)E(s)$$

$$E(s) = R(s) - C(s)H(s) = R(s) - G(s)E(s)H(s)$$

$$E(s)(1 + G(s)H(s)) = R(s)$$

$$E(s)= \frac{R(s)}{(1 +G(s)H(s))}$$

$$e_{ss} = \lim_{s\to0} sE(s) =\lim_{s\to0} \frac{sR(s)}{(1 + G(s)H(s))}$$

When a control system is excited with standard input signal, the steady state error may be zero, constant or infinity. Its value depends on the type number and input signal.

   a) Type-0 system will have a constant steady state error when the input is step signal

   b) Type-1 system will have a constant steady state error when the input is ramp signal

   c) Type-2 system will have a constant steady state error when the input is parabolic signal

$$\text{For unit step input, } e_{ss} = \frac{1}{1+K_p}$$

$$\text{For unit ramp input, } e_{ss} = \frac{1}{K_v}$$

$$\text{For unit parabolic input, } e_{ss} = \frac{1}{K_a}$$

Static error constants for various type number of systems

Error constants	Type number of system			
	0	**1**	**2**	**3**
K_p	Constant	∞	∞	∞
K_v	0	Constant	∞	∞
K_a	0	0	Constant	∞

Steady state error for various types of input

Input signal	Type number of system			
	0	**1**	**2**	**3**
K_p	$\dfrac{1}{1+K_p}$	0	0	0
K_v	∞	$\dfrac{1}{K_v}$	0	0
K_a	∞	∞	$\dfrac{1}{K_a}$	0

2.2 TIME DOMAIN SPECIFICATIONS

The desired performance characteristics of control systems are specified in terms of time domain specifications. Systems with energy storage elements cannot respond instantaneously and will exhibit transient responses, whenever they are subjected to inputs or disturbances. The desired performance characteristics of a system of any order may be specified in terms of the transient response to a unit step input signal. The response of a second order system for unit step input with various values of damping ratio is shown in figure 2.2.1.
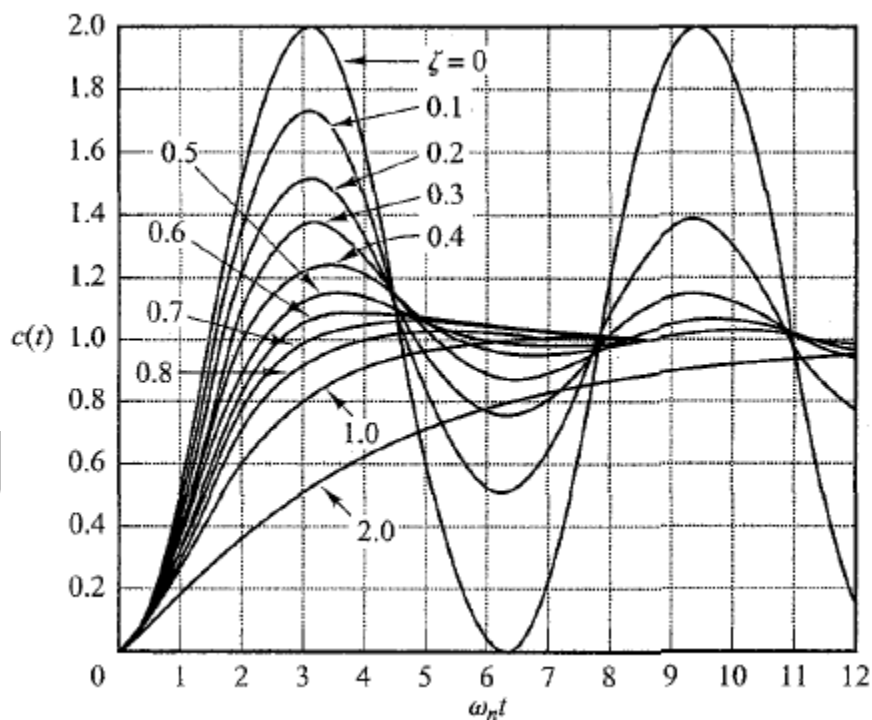
Figure 2.2.1 Time Response

[Source: "Modern Control Engineering" by Katsuhiko Ogata, Page: 229]

The transient response of a system to a unit step input depends on the initial conditions. Therefore, to compare the time response of various systems it is necessary to start with standard initial conditions. The most practical standard is to start with the system at rest and so output and all time derivatives before t=0 will be zero. The transient response of a practical control system often exhibits damped oscillation before reaching steady state. A typical damped oscillatory response of a system is shown in figure 2.2.2.
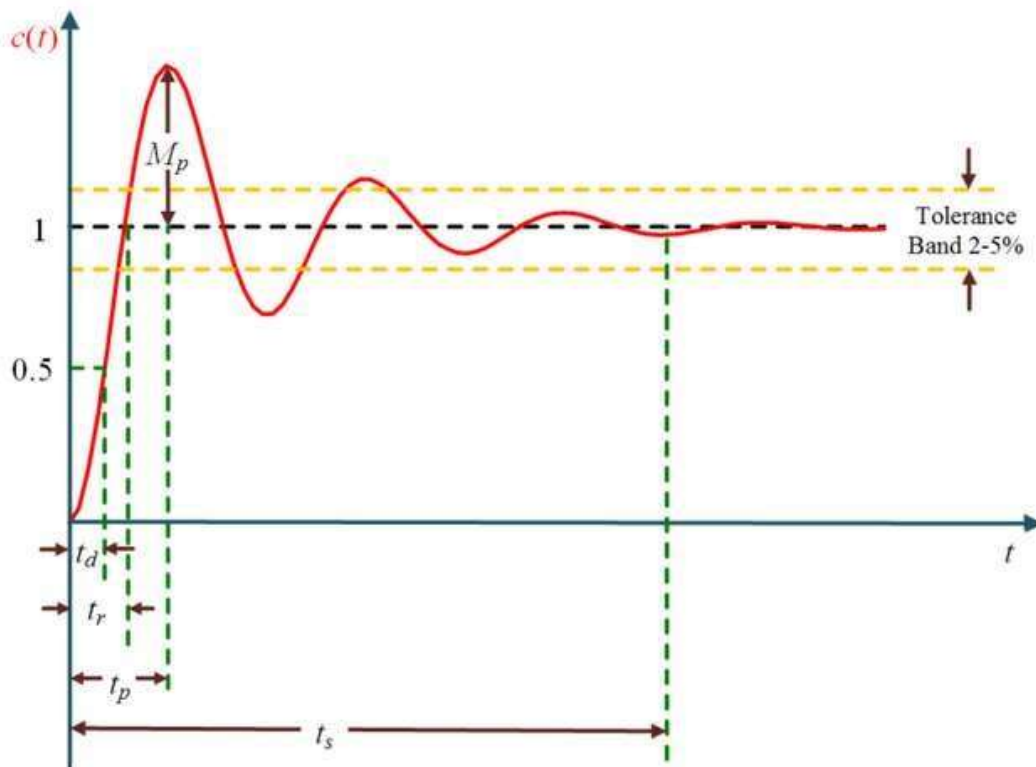
Figure 2.2.2 Transient and steady-state response analyses

[Source: "Modern Control Engineering" by Katsuhiko Ogata, Page: 230]

The transient response characteristics of a control system to a unit step input is specified in terms of the following time domain specifications:

1. Delay time, t_d: It is the time required for the response to reach 50% of the steady state value for the first time.

$$t_d = \frac{1+0.7\zeta}{\omega_n}$$

2. Rise time, t_r: It is the time required for the response to reach 100% of the steady state value for under damped systems. However, for over damped systems, it is taken as the time required for the response to rise from 10% to 90% of the steady state value. The unit step response of second order system for underdamped case is given by,

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{(1-\zeta^2)}}\sin(\omega_d t + \theta)$$

At $t = t_r$, $c(t) = c(t_r) = 1$

$$c(t_r) = 1 - \frac{e^{-\zeta\omega_n t_r}}{\sqrt{(1-\zeta^2)}}\sin(\omega_d t_r + \theta) = 1$$

$$\frac{-e^{-\zeta\omega_n t_r}}{\sqrt{(1-\zeta^2)}}\sin(\omega_d t_r + \theta) = 0$$

Since $-e^{-\zeta\omega_n t_r} \neq 0$, the term, $\sin(\omega_d t_r + \theta) = 0$, When

$\Phi = 0, \pi, 2\pi, 3\pi, \ldots$ $\qquad\qquad$ $\sin \Phi = 0$

$$\omega_d t_r + \theta = \pi$$

$$\omega_d t_r = \pi - \theta$$

$$t_r = \frac{\pi - \theta}{\omega_d}$$

On constructing right angled triangle,

$$\tan\theta = \frac{\sqrt{(1-\zeta^2)}}{\zeta}$$

$$\theta = \tan^{-1}\frac{\sqrt{(1-\zeta^2)}}{\zeta}$$

$$Damped\ frequency,\ \omega_d = \omega_n\sqrt{(1-\zeta^2)}$$

$$t_r = \frac{\pi - \tan^{-1}\left(\frac{\sqrt{(1-\zeta^2)}}{\zeta}\right)}{\omega_n\sqrt{(1-\zeta^2)}}$$

3. Peak time, t_p: It is the time required for the response to reach the maximum or peak value of the response. To find the expression for peak time, t_p, differentiate c(t) with respect to 't' and equate to zero.

$$\frac{d}{dt}c(t)\big|_{t=t_p} = 0$$

The unit step response of under damped second order system is given by

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{(1-\zeta^2)}}\sin(\omega_d t + \theta)$$

Differentiating c(t) with respect to 't',

$$\frac{d}{dt}c(t) = \frac{-e^{-\zeta\omega_n t}}{\sqrt{(1-\zeta^2)}}(-\zeta\omega_n)\sin(\omega_d t + \theta) + \left(\frac{-e^{-\zeta\omega_n t}}{\sqrt{(1-\zeta^2)}}\right)\cos(\omega_d t + \theta)\omega_d$$

Put $\omega_d = \omega_n \sqrt{(1 - \zeta^2)}$,

$$\frac{d}{dt} c(t) = \frac{e^{-\zeta \omega_n t}}{\sqrt{(1-\zeta^2)}} (\zeta \omega_n) \sin(\omega_d t + \theta) - \left(\frac{e^{-\zeta \omega_n t}}{\sqrt{(1-\zeta^2)}} \right) \cos(\omega_d t + \theta) \omega_n \sqrt{(1 - \zeta^2)}$$

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$$\frac{d}{dt} c(t) = \frac{e^{-\zeta \omega_n t}}{\sqrt{(1-\zeta^2)}} (\zeta \omega_n) \sin(\omega_d t + \theta) - \left(\frac{e^{-\zeta \omega_n t}}{\sqrt{(1-\zeta^2)}} \right) \cos(\omega_d t + \theta) \omega_n \sqrt{(1 - }$$

$$= \frac{\omega_n e^{-\zeta \omega_n t}}{\sqrt{(1-\zeta^2)}}[\zeta \sin(\omega_d t+\theta)-(\sqrt{(1-\zeta^2)})\cos(\omega_d t+\theta)]$$

$$= \frac{\omega_n e^{-\zeta \omega_n t}}{\sqrt{(1-\zeta^2)}}[\cos\theta \sin(\omega_d t+\theta) - \sin\theta \cos(\omega_d t+\theta)]$$

$$= \frac{\omega_n e^{-\zeta \omega_n t}}{\sqrt{(1-\zeta)^2}}[\sin(\omega_d t+\theta-\theta)]$$

$$= \frac{\omega_n e^{-\zeta \omega_n t}}{\sqrt{(1-\zeta)^2}}[\sin(\omega_d t)]$$

At $t = t_p, \frac{d}{dt}c(t)=0$

$$\frac{\omega_n e^{-\zeta \omega_n t_p}}{\sqrt{(1-\zeta^2)}}[\sin(\omega_d t_p)]=0$$

Since, $e^{-\zeta \omega_n t_p} \neq 0$, the term, $[\sin(\omega_d t_p)]=0$

When $\Phi = 0, \pi, 2\pi, 3\pi,\ldots$ $\qquad \sin \Phi = 0$

$$\omega_d t_p = \pi$$

$$t_p = \frac{\pi}{\omega_d}$$

On substituting, we get,

$$t_p = \frac{\pi}{\omega_n \sqrt{(1-\zeta^2)}}$$

4. Peak overshoot, M_p: It is defined as the difference between the peak value of the response and the steady state value. Iris usually expressed in percent of the steady state value. If the time for the peak is t_p, percent peak overshoot is given by,

$$\text{Maximum percent overshoot} = \frac{c(t_p)-c(\infty)}{c(\infty)}$$

$$At \, t=\infty, c(t)=c(\infty)=1- \frac{e^{-\zeta \omega_n \infty}}{\sqrt{(1-\zeta^2)}}\sin(\omega_d t+\theta)=1-0=1$$

$$At \, t=t_p, c(t)=c(t_p)=1- \frac{e^{-\zeta \omega_n t_p}}{\sqrt{(1-\zeta^2)}}\sin(\omega_d t_p+\theta)$$

$$=1-\frac{e^{-\zeta \omega_n \frac{\pi}{\omega_n\sqrt{(1-\zeta^2)}}}}{\sqrt{(1-\zeta^2)}}\sin\left(\omega_d \frac{\pi}{\omega_d}+\theta\right)$$

$$=1-\frac{e^{-\zeta \frac{\pi}{\sqrt{(1-\zeta^2)}}}}{\sqrt{(1-\zeta^2)}}\sin(\pi+\theta)$$

$$= 1 - \frac{e^{-\zeta \frac{\pi}{\sqrt{(1-\zeta^2)}}}}{\sqrt{(1-\zeta^2)}} \sin(\theta) = 1 + \frac{e^{\frac{\zeta\pi}{\sqrt{(1-\zeta^2)}}}}{\sqrt{(1-\zeta^2)}} \sqrt{(1-\zeta^2)}$$

$$\% M_p = \frac{c(t_p) - c(\infty)}{c(\infty)}$$

$$\% M_p = e^{\frac{\pi\zeta}{\sqrt{(1-\zeta^2)}}} \times 100$$

5. Settling time, t_s: It is the time taken by the response to reach and stay within a specified error. It is usually expressed as percentage of final value. The usual tolerable error is 2% and 5% of the final value.

The response of second order system has two components. They are

a. Decaying exponential component, $\dfrac{e^{-\zeta\omega_n t}}{\sqrt{(1-\zeta^2)}}$

b. Sinusoidal component, $\sin(\omega_d t + \theta)$

In these terms, the decaying component term dampens or reduces the oscillations produced by sinusoidal component. Hence, the settling time is decided by the exponential component. The settling time can be found out by equating exponential component to percentage tolerance errors.

For 2% tolerance error band, at t = t$_s$, $\dfrac{e^{-\zeta\omega_n t_s}}{\sqrt{(1-\zeta^2)}} = 0.02$

For least values of ζ, $e^{-\zeta\omega_n t_s} = 0.02$

On taking natural logarithm on both sides, we get,

$$-\zeta\omega_n t_s = \ln(0.02) = -4 \quad 4$$
$$t_s = \frac{4}{\zeta\omega_n} = 4T$$

For 5% tolerance error band, at t = t$_s$, $\dfrac{e^{-\zeta\omega_n t_s}}{\sqrt{(1-\zeta^2)}} = 0.05$

For least values of ζ, $e^{-\zeta\omega_n t_s} = 0.05$

On taking natural logarithm on both sides, we get,

$$-\zeta\omega_n t_s = \ln(0.02) = -3 \quad 3$$
$$t_s = \frac{3}{\zeta\omega_n} = 3T$$

$$Settling\ time, t_s = \frac{4}{\zeta\omega_n}\ for\ 2\%\ error$$

$$Settling\ time, t_s = \frac{3}{\zeta\omega_n}\ for\ 5\%\ error$$

The performance of a system is usually evaluated in terms of the following qualities:

- How fast it is able to respond to the input?
- How fast it is reaching the desired output?
- What is the error between the desired output and the actual output, once the transients die down and steady state is achieved?
- Does it oscillate around the desired value?
- Is the output continuously increasing with time or is it bounded?
- These are the specifications to be given for the design of a controller for a given system.

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2.9 TIME RESPONSE ANALYSIS

- Two types of inputs can be applied to a control system
- Command Input or Reference Input $y_r(t)$
- Disturbance Input $w(t)$

*(*External disturbances $w(t)$ are typically uncontrolled variations in the load on a control system). In systems controlling mechanical motions, load disturbances may represent forces. In voltage regulating systems, variations in electrical load area major source of disturbances.

In general, the closed loop transfer function of a system is denoted as M(s).

$$M(s) = \frac{b_0 s^M + b_1 s^{M-1} + b_2 s^{M-2} + \ldots\ldots + b_{M-1} s + b_M}{a_0 s^N + a_1 s^{N-1} + a_2 s^{N-2} + \ldots\ldots + a_{N-1} s + a_N}$$

2.1 TIME RESPONSE

The time response of the system is the output of the closed loop system as a function of time. It describes the behavior of a system and contains much information about it with respect to time response specification. Time response is formed by the transient response and the steady state response.

Time response = Transient response + Steady state response

Transient time response

Transient response (Natural response) describes the behavior of the system in its first short time until arrives the steady state value and this response will be our study focus. If the input is step function then the output or the response is called step time response and if the input is ramp, the response is called ramp time response, etc.

$$y(t) = y_{tr}(t) + y_{ss}(t)$$

The transient response is defined as the part of the time response that goes to zero as time becomes very large. Thus $y_t(t)$ has the property

$$\text{Lim } y_t(t) = 0, t \rightarrow \infty$$

The time required to achieve the final value is called transient period. The transient response may be exponential or oscillatory in nature. Output response consists of the sum of forced response (form the input) and natural response (from the nature of the system).The transient response is the change in output response from the beginning of the response to the final state of the response and the steady state response is the output response as time is approaching infinity (or no more changes at the output).

Steady State Response

The steady state response is the part of the total response that remains after the transient has died out. For a position control system, the steady state response when compared to with the desired reference position gives an indication of the final accuracy of the system. If the steady state response of the output does not agree with the desired reference exactly, the system is said to have steady state error.

2.3   TYPES OF TEST INPUT

The knowledge of input signal is required to predict the response of a system. In most of the systems, the input signals are not known ahead of time and also it is difficult to express the input signals mathematically by simple equations. The characteristics of actual input signals are a sudden shock, a sudden change, a constant velocity and a constant acceleration. Hence test signals which resembles these characteristics are used as input signals to predict the performance of the system. The commonly use test input signals are impulse, step, ramp, acceleration and sinusoidal signals.

Standard Input Signals

1. Step signal   2. Unit step signal

3. Ramp signal   4. Unit ramp signal

5. Parabolic signal   6. Unit parabolic signal

7. Impulse signal   8. Sinusoidal signal

STEP SIGNAL

The step signal is a signal whose value changes from zero to A at t=0 and remains constant at A for t>0. The step signal resembles an actual steady input to a system. A special case of step signal is unit step in which A is unity.

RAMP SIGNAL

The ramp signal is a signal whose value increases linearly with time from an initial value of zero at t=0. The ramp signal resembles a constant velocity input to the system. A special case of ramp signal is unit ramp signal in which the value of A is unity.
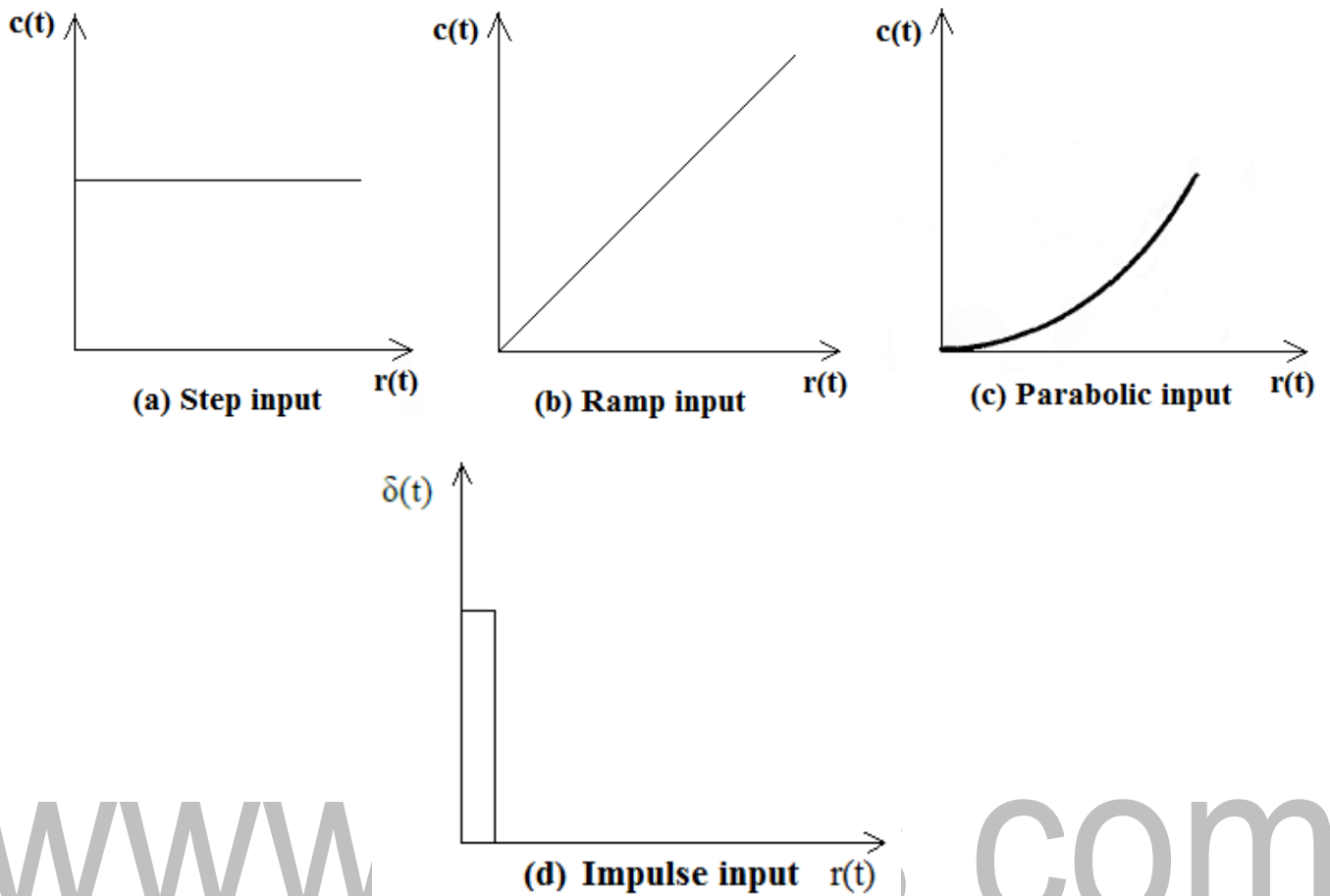
PARABOLIC SIGNAL

In parabolic signal, the instantaneous value varies as square of the time from an initial value of zero at t=0. The sketch of the signal with respect to time resembles a parabola. The parabolic signal resembles a constant acceleration input to the system. A special case of parabolic signal is unit parabolic signal in which A is unity.

IMPULSE SIGNAL

A signal of very large magnitude which is available for very short duration is called impulse signal. Ideal impulse signal is a signal with infinite magnitude and zero duration but with an area of A. The unit impulse signal is a special case, in which A is unity. Since perfect impulse cannot be achieved in practice, it is usually approximated by a pulse of

small width but with area, A. Mathematically an impulse signal is the derivative of a step signal. Laplace transform of the impulse function is unity.

Figure 2.3.1 Standard test signals

[Source: "Control Systems Engineering" by I J Nagrath, M Gopal, Page: 196]

Input	r(t)	R(s)
Step input	A	A/s
Ramp input	At	A/s^2
Parabolic input	$At^2/2$	A/s^3
Impulse input	$\delta(t)$	1

Input	Function	Description	Sketch	Use
Impulse	$\delta(t)$	$\delta(t) = \infty$ for $0- < t < 0+$ $= 0$ elsewhere $\int_{0-}^{0+} \delta(t)\,dt = 1$		Transient response Modeling
Step	$u(t)$	$u(t) = 1$ for $t > 0$ $= 0$ for $t < 0$		Transient response Steady-state error
Ramp	$tu(t)$	$tu(t) = t$ for $t \geq 0$ $= 0$ elsewhere		Steady-state error
Parabola	$\frac{1}{2}t^2 u(t)$	$\frac{1}{2}t^2 u(t) = \frac{1}{2}t^2$ for $t \geq 0$ $= 0$ elsewhere		Steady-state error
Sinusoid	$\sin \omega t$			Transient response Modeling Steady-state error