Behaviour of Plane waves at the inteface of two media:

We consider ε, μ, σ the propagation of uniform plane waves in an unbounded homogeneous medium. In practice, the wave will propagate in bounded regions where several values of will be present. When plane wave travelling in one medium meets a different medium, it is partly reflected and partly transmitted. In this section, we consider wave reflection and transmission at planar boundary between two media as shown in figure 1.1.

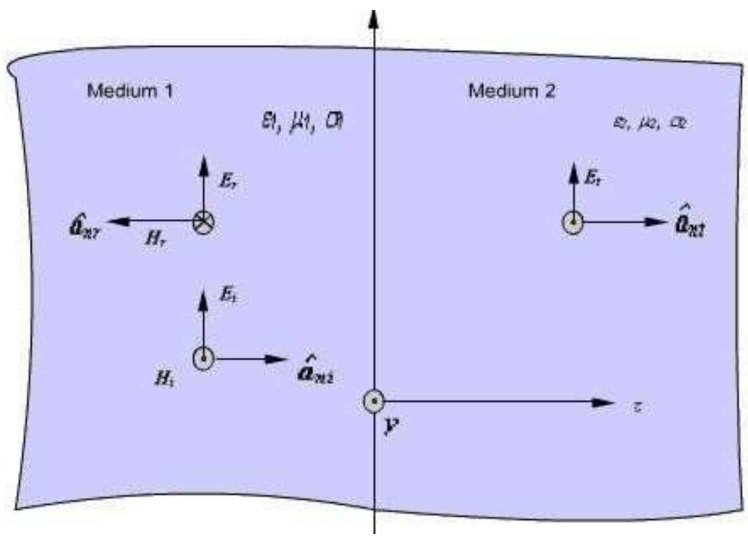

Fig 1.1 : Normal Incidence at a plane boundary
(www.brainkart.com/subject/Electromagnetic-Theory_206/)

Case1: Let z = 0 plane represent the interface between two media.
Medium 1is $(\varepsilon_1, \mu_1, \sigma_1)$ and medium 2 is characterized by $(\varepsilon_2, \mu_2, \sigma_2)$.

Let the subscripts '*i*' denotes incident, '*r*' denotes reflected and '*t*' denotes transmittedfield components respectively.

The incident wave is assumed to be a plane wave polarized along *x* and travelling inmedium

1 along $\hat{a}_z$ direction. From equation (5.24) we can write

$$\vec{E}_i(z) = E_{io} e^{-\gamma_1 z} \hat{a}_x \quad \text{..................(5.49.a)}$$

$$\vec{H}_i(z) = \frac{1}{\eta_i} \hat{a}_z \times E_{io}(z) = \frac{E_o}{\eta_i} e^{-\gamma_1 z} \hat{a}_y \quad \text{......................(5.49.b)}$$

where $\gamma_1 = \sqrt{j\omega\mu_1(\sigma_1 + j\omega\varepsilon_1)}$ and $\eta_1 = \sqrt{\dfrac{j\omega\mu_1}{\sigma_1 + j\omega\varepsilon_2}}$.

Because of the presence of the second medium at z =0, the incident wave will undergo partial reflection and partial transmission.
The reflected wave will travel along $\hat{a}_z$ in medium

1. The reflected field components are:

$$\vec{E}_r = E_{ro} e^{\gamma_1 z} \hat{a}_x \text{ ....(5.50a)}$$

$$\vec{H}_r = \frac{1}{\eta_1}\left(-\hat{a}_z\right) \times E_{ro} e^{\gamma_1 z} \hat{a}_x = -\frac{E_{ro}}{\eta_1} e^{\gamma_1 z} \hat{a}_y \text{ ....(5.50b)}$$

The transmitted wave will travel in medium 2 along $\hat{a}_z$ for which the field components are

$$\vec{E}_t = E_{t0} e^{-\gamma_2 z} \hat{a}_x \text{ ....(5.51a)}$$

$$\vec{H}_t = \frac{E_{t0}}{\eta_2} e^{-\gamma_2 z} \hat{a}_y \text{ ....(5.51b)}$$

where $\gamma_2 = \sqrt{j\omega\mu_2(\sigma_2 + j\omega\varepsilon_2)}$ and $\eta_2 = \sqrt{\dfrac{j\omega\mu_2}{\sigma_2 + j\omega\varepsilon_2}}$

In medium 1, $\vec{E}_1 = \vec{E}_i + \vec{E}_r$ and $\vec{H}_1 = \vec{H}_i + \vec{H}_r$

and in medium 2, $\vec{E}_2 = \vec{E}_t$ and $\vec{H}_2 = \vec{H}_t$

Applying boundary conditions at the interface z = 0, i.e., continuity of tangential field components and noting that incident, reflected and transmitted field components are tangential at the boundary, we can write

$$\vec{E}_i(0) + \vec{E}_r(0) = \vec{E}_t(0)$$

$$\& \ \vec{H}_i(0) + \vec{H}_r(0) = \vec{H}_t(0)$$

From equation 5.49 to 5.51 we get,

$$E_{io} + E_{ro} = E_{to} \qquad \text{.......(5.52a)}$$

$$\frac{E_{io}}{\eta_1} - \frac{E_{ro}}{\eta_1} = \frac{E_{to}}{\eta_2} \qquad \text{.......(5.52b)}$$

Eliminating
Eto ,

$$\frac{E_{io}}{\eta_1} - \frac{E_{ro}}{\eta_1} = \frac{1}{\eta_2}\left(E_{io} + E_{ro}\right)$$

$$\text{or, } E_{io}\left(\frac{1}{\eta_1} - \frac{1}{\eta_2}\right) = E_{ro}\left(\frac{1}{\eta_1} + \frac{1}{\eta_2}\right)$$

$$\text{or, } E_{ro} = \tau E_{io}$$

...............(5.53)

$$\tau = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$$

is called the reflection coefficient.

From equation (5.52), we can

write

$$2E_{io} = E_{to}\left[1 + \frac{\eta_1}{\eta_2}\right]$$

$$\text{or, } T = \frac{2\eta_2}{\eta_1 + \eta_2} \qquad \text{.......(5.54)}$$

$$E_{to} = \frac{2\eta_2}{\eta_1 + \eta_2} E_{io} = TE_{io}$$

is called the transmission

coefficient. We observe that,

$$T = \frac{2\eta_2}{\eta_1 + \eta_2} = \frac{\eta_2 - \eta_1 + \eta_1 + \eta_2}{\eta_1 + \eta_2} = 1 + \tau \qquad \text{.......(5.55)}$$

The following may be noted

Let us now consider specific cases:

Case I: Normal incidence on a plane conducting boundary

The medium 1 is perfect dielectric $(\sigma_1 = 0)$ and medium 2 is perfectly conducting $(\sigma_2 = \infty)$.

$$\therefore \eta_1 = \sqrt{\frac{\mu_1}{\epsilon_1}}$$

$$\eta_2 = 0$$

$$\gamma_1 = \sqrt{(j\omega\mu_1)(j\omega\epsilon_1)}$$
$$= j\omega\sqrt{\mu_1\epsilon_1} = j\beta_1$$

From (5.53) and (5.54)

$$\tau = -1$$

and T =0

Hence the wave is not transmitted to medium 2, it gets reflected entirely from the interface to the medium 1.

$$\therefore \vec{E}_1(z) = E_{io}e^{-j\beta_1 z}\,\hat{a}_x - E_{io}e^{j\beta_1 z}\,\hat{a}_x = -2jE_{io}\sin\beta_1 z\,\hat{a}_x$$

&

$$\therefore \vec{E}_1(z,t) = \mathrm{Re}\left[-2jE_{io}\sin\beta_1 z e^{j\omega t}\right]\hat{a}_x = 2E_{io}\sin\beta_1 z\sin\omega t\,\hat{a}_x \quad \text{.................................(5.55)}$$

Proceeding in the same manner for the magnetic field in region 1, we can show that,

$$\vec{H}_1(z,t) = \hat{a}_y\frac{2E_{io}}{\eta_1}\cos\beta_1 z\cos\omega t \quad \text{..(5.57)}$$

The wave in medium 1 thus becomes a **standing wave** due to the super position of a forward travelling wave and a backward travelling wave. For a given ' t', both $\vec{E}_1$ and $\vec{H}_1$ vary sinusoidally with distance measured from z = 0. This is shown in figure 1.2.

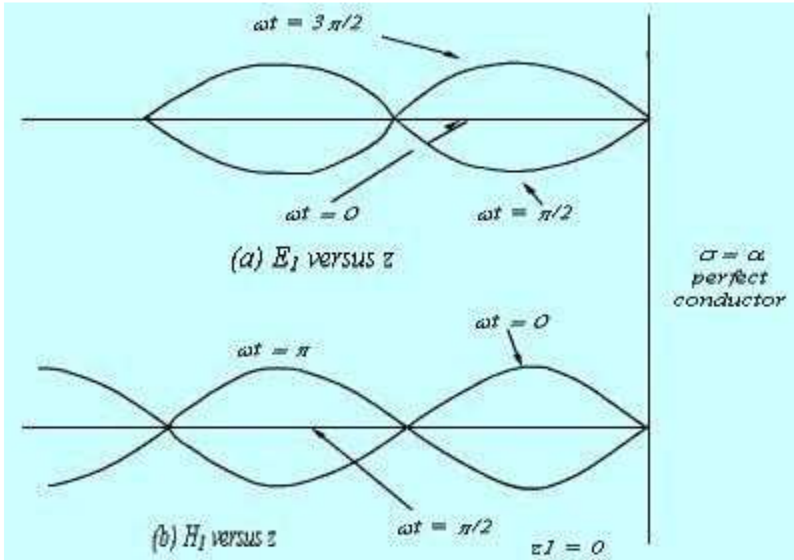

Figure 1.2: Generation of standing wave
(www.brainkart.com/subject/Electromagnetic-Theory_206/)

Zeroes of $E_1(z,t)$ and Maxima of $H_1(z,t)$.

Maxima of $E_1(z,t)$ and zeroes of $H_1(z,t)$.

$$\left.\begin{array}{l} \text{occur at } \beta_1 z = -n\pi \quad \text{or } z = -n\dfrac{\lambda}{2} \\ \text{occur at } \beta_1 z = -(2n+1)\dfrac{\pi}{2} \quad \text{or } z = -(2n+1)\dfrac{\lambda}{4}, \quad n = 0,1,2,.... \end{array}\right\} \quad ....(5.58)$$

Case2: Normal incidence on a plane dielectric boundary

If the medium 2 is not a perfect conductor (i.e. $\sigma_2 \neq \infty$) partial reflection will result. There will be a reflected wave in the medium 1 and a transmitted wave in the medium 2.Because of the reflected wave, standing wave is formed in medium 1.
From equation (5.49(a)) and equation (5.53) we can write

$$\vec{E}_1 = E_{io}\left(e^{-\gamma_1 z} + \Gamma e^{\gamma_1 z}\right)\hat{a}_x \quad ....(5.59)$$

Let us consider the scenario when both the media are dissipation less i.e. perfect dielectrics ($\sigma_1 = 0, \sigma_2 = 0$)

$$\gamma_1 = j\omega\sqrt{\mu_1\varepsilon_1} = j\beta_1 \qquad \eta_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}}$$

$$\gamma_2 = j\omega\sqrt{\mu_2\varepsilon_2} = j\beta_2 \qquad \eta_2 = \sqrt{\frac{\mu_2}{\varepsilon_2}} \quad ....(5.50)$$

In this case both η_1 and η_1 become real numbers.

$$\begin{aligned}\vec{E}_1 &= \hat{a}_x E_{io}\left(e^{-j\beta_1 z} + \Gamma e^{j\beta_1 z}\right)\\ &= \hat{a}_x E_{io}\left((1+T)e^{-j\beta_1 z} + \Gamma\left(e^{j\beta_1 z} - e^{-j\beta_1 z}\right)\right)\\ &= \hat{a}_x E_{io}\left(Te^{-j\beta_1 z} + \Gamma\left(2j\sin\beta_1 z\right)\right) \quad ....(5.51)\end{aligned}$$

From (5.51), we can see that, in medium 1 we have a traveling wave component with amplitude TEio and a standing wave component with amplitude 2JEio.

The location of the maximum and the minimum of the electric and magnetic field components in the medium 1from the interface can be found as follows.

The electric field in medium 1 can be written as

$$\vec{E}_1 = \hat{a}_x E_{io} e^{-j\beta_1 z}\left(1 + \Gamma e^{j2\beta_1 z}\right) \quad ....(5.52)$$

If $\eta_2 > \eta_1$ i.e. $\Gamma > 0$

The maximum value of the electric field is

$$\left|\vec{E}_1\right|_{max} = E_{io}\left(1+T\right) \quad ....(5.53)$$

and this occurs when

$$2\beta_1 z_{max} = -2n\pi$$

or $$z_{max} = -\frac{n\pi}{\beta_1} = -\frac{n\pi}{2\pi/\lambda_1} = -\frac{n}{2}\lambda_1$$, n = 0, 1, 2, 3 ....(5.54)

The minimum value of $|\vec{E}_1|$ is

$$|\vec{E}_1|_{min} = E_{io}(1-\Gamma)$$ ....(5.55)

And this occurs when

$$2\beta_1 z_{min} = -(2n+1)\pi$$

or $$z_{min} = -(2n+1)\frac{\lambda_1}{4}$$, n = 0, 1, 2,

Γ $\eta_2 < \eta_1$ i.e. <3............................ (5.55) For

The maximum value of $|\vec{E}_1|$ is $E_{io}(1-\Gamma)$ which occurs at the z_{min} locations and the minimum

value of $|\vec{E}_1|$ is $E_{io}(1+\Gamma)$ which occurs at z_{max} locations as given by the equations (5.54) and (5.55).

From our discussions so far we observe that $\frac{|E|_{max}}{|E|_{min}}$ can be written as

$$S = \frac{|E|_{max}}{|E|_{min}} = \frac{1+|\Gamma|}{1-|\Gamma|}$$ ....(5.57)

The quantity *S* is called as the standing wave ratio.

As $0 \le |\Gamma| \le 1$ the range of *S* is given by $1 \le S \le \infty$

From (5.52), we can write the expression for the magnetic field in medium 1 as

$$\vec{H}_1 = \hat{a}_y \frac{E_{io}}{\eta_1} e^{-j\beta_1 z}\left(1-\Gamma e^{j2\beta_1 z}\right)$$ ....(5.58)

From (5.58) we find that $|\vec{H}_1|$ will be maximum at locations where $|\vec{E}_1|$ is minimum and vice versa.
In medium 2, the transmitted wave propagates in the + z direction.

Oblique Incidence of EM wave at an interface

So far we have discuss the case of normal incidence where electromagnetic wave traveling in a lossless medium impinges normally at the interface of a second medium. In this section we shall consider the case of oblique incidence. As before, we consider two cases

i. When the second medium is a perfect conductor. ii. When the second medium is a perfect dielectric.

A plane incidence is defined as the plane containing the vector indicating the direction of propagation of the incident wave and normal $\vec{E}_i$ to the interface. We study two specific cases when the incident electric field $\vec{E}_i$ is perpendicular to the plane of incidence (perpendicular polarization) and is parallel to the plane of incidence (parallel polarization). For a general case, the incident wave may have arbitrary polarization but the same can be expressed as a linear combination of these two individual cases.

Oblique Incidence at a plane conducting boundary i. Perpendicular Polarization
The situation is depicted in figure 1.3.

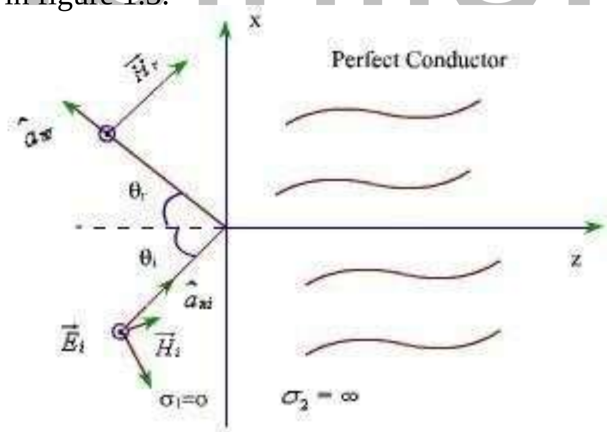

Figure1.3:Perpendicularpolarization
(www.brainkart.com/subject/Electromagnetic-Theory_206/)

As the EM field inside the perfect conductor is zero, the interface reflects the incident plane wave. $\hat{a}_{ni}$ and $\hat{a}_{nr}$ respectively represent the unit vector in the direction of propagation of the incident and reflected waves, θ_i is the angle of incidence and θ_r is the angle of reflection.

We find that

$$\hat{a}_{ni} = \hat{a}_z \cos\theta_i + \hat{a}_x \sin\theta_i$$
$$\hat{a}_{nr} = -\hat{a}_z \cos\theta_r + \hat{a}_x \sin\theta_r \quad \text{.............................(5.59)}$$

Since the incident wave is considered to be perpendicular to the plane of incidence, which for the present case happens to be xz plane, the electric field has only y-component.

$$\vec{E}_i(x,z) = \hat{a}_y E_{io} e^{-j\beta_1 \bar{a}_{ni}\cdot\bar{r}}$$
$$= \hat{a}_y E_{io} e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

The corresponding magnetic field is given by

$$\vec{H}_i(x,z) = \frac{1}{\eta_1}\left[\hat{a}_n \times \vec{E}_i(x,z)\right]$$
$$= \frac{1}{\eta_1}\left[-\cos\theta_i \hat{a}_x + \sin\theta_i \hat{a}_z\right] E_{io} e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)} \quad \text{..........................(5.70)}$$

Similarly, we can write the reflected waves as

$$\vec{E}_r(x,z) = \hat{a}_y E_{ro} e^{-j\beta_1 \bar{a}_{nr}\cdot\bar{r}}$$
$$= \hat{a}_y E_{ro} e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)} \quad \text{..(5.71)}$$

Since at the interface z=o, the tangential electric field is zero.

$$E_{io} e^{-j\beta_1 x\sin\theta_i} + E_{ro} e^{-j\beta_1 x\sin\theta_r} = 0 \quad \text{..(5.72)}$$

Consider in equation (5.72) is satisfied if we have

$$E_{ro} = -E_{io}$$
$$\text{and } \theta_i = \theta_r \quad \text{.................................(5.73)}$$

The condition $\theta_i = \theta_r$ is Snell's law of reflection.

$$\therefore \vec{E}_r(x,z) = -\hat{a}_y E_{io} e^{-j\beta_1(x\sin\theta_i - z\cos\theta_i)} \quad \text{.................................(5.74)}$$

The total electric field is given by

$$\vec{E}_1(x,z) = \vec{E}_i(x,z) + \vec{E}_r(x,z)$$

$$= -\hat{a}_y 2jE_{io} \sin(\beta_1 z \cos\theta_i) e^{-j\beta_1 x \sin\theta_i} \quad \text{.................................(5.75)}$$

and $\vec{H}_r(x,z) = \dfrac{1}{\eta_1}\left[\hat{a}_{nr} \times \vec{E}_r(x,z)\right]$ ....(5.76)

Sim $= \dfrac{E_{io}}{\eta_1}\left[-\hat{a}_x \cos\theta_i - \hat{a}_z \sin\theta_i\right] e^{-j\beta_1(x\sin\theta_i - z\cos\theta_i)}$

............................(5.77)

$$\vec{H}_1(x,z) = -2\frac{E_{io}}{\eta_1}\left[\hat{a}_x \cos\theta_i \cos(\beta_1 z\cos\theta_i) e^{-j\beta_1 x\sin\theta_i} + \hat{a}_z j \sin\theta_i \sin(\beta_1 z \cos\theta_i) e^{-j\beta_1 x \sin\theta_i}\right]$$

.............................(5.78)

The wave propagating along the x direction has its amplitude varying with z and hence constitutes a **non uniform** plane wave. Further, only electric field $\vec{E}_1$ is perpendicular to the direction of propagation (i.e. x), the magnetic field has component along the direction of propagation. Such waves are called transverse electric or TE waves.

ii. **Parallel Polarization**:

In this case also $\hat{a}_{ni}$ and $\hat{a}_{nr}$ are given by equations (5.59). Here $\vec{H}_1$ and $\vec{H}_r$ have only y component.

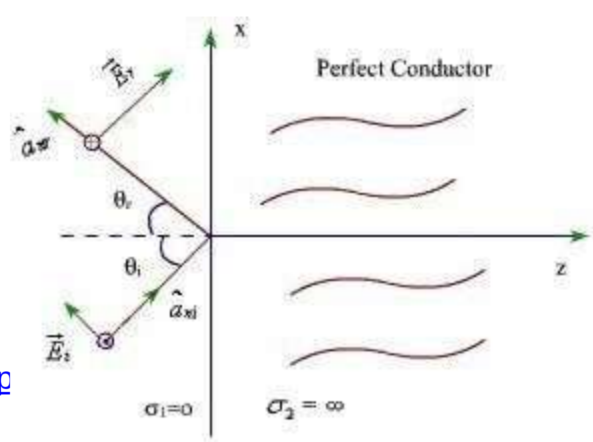

Figure 1.4: Parallel *Polarization*

(www.brainkart.com/subject/Electromagnetic-Theory_206/)

With reference to fig (1.4), the field components can be written

as: Incident field components:

$$\vec{E}_i(x,z) = E_{io}\left[\cos\theta_i \hat{a}_x - \sin\theta_i \hat{a}_z\right] e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

$$\vec{H}_i(x,z) = \hat{a}_y \frac{E_{io}}{\eta_1} e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

............................(5.79)

Reflected field components:

$$\vec{E}_r(x,z) = E_{ro}\left[\hat{a}_x \cos\theta_r + \hat{a}_z \sin\theta_r\right] e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

$$\vec{H}_r(x,z) = -\hat{a}_y \frac{E_{ro}}{\eta_1} e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

............................(5.80)

Since the total tangential electric field component at the interface is zero.

$$E_i(x,0) + E(x,0) = 0$$

Which leads to $E_{io} = -E_{ro}$ and $\theta_i = \theta_r$ as before.

Substituting these quantities in (5.79) and adding the incident and reflected electric and magnetic field components the total electric and magnetic fields can be written as

$$\vec{E}_i(x,z) = -2E_{io}\left[\hat{a}_x j\cos\theta_i \sin(\beta_1 z\cos\theta_i) + \hat{a}_z \sin\theta_i \cos(\beta_1 z\cos\theta_i)\right] e^{-j\beta_1 x\sin\theta_i}$$

$$\text{and}\quad \vec{H}_i(x,z) = \hat{a}_y \frac{2E_{io}}{\eta_1}\cos(\beta_1 z\cos\theta_i) e^{-j\beta_1 x\sin\theta_i}$$

...........................(5.81)

Once again, we find a standing wave pattern along z for the x and y components of $\vec{E}$ and $\vec{H}$, while a non uniform plane wave propagates along x with a phase velocity given $v_{p1x} = \frac{v_{p1}}{\sin\theta_i}$ by where $v_{p1} = \frac{\omega}{\beta_1}$. Since, for this propagating wave, magnetic field is in transverse direction, such waves are called transverse magnetic or TM waves.

Oblique incidence at a plane dielectric interface

We continue our discussion on the behavior of plane waves at an interface; this time we consider a plane dielectric interface. As earlier, we consider the two specific cases, namely parallel and perpendicular polarization.

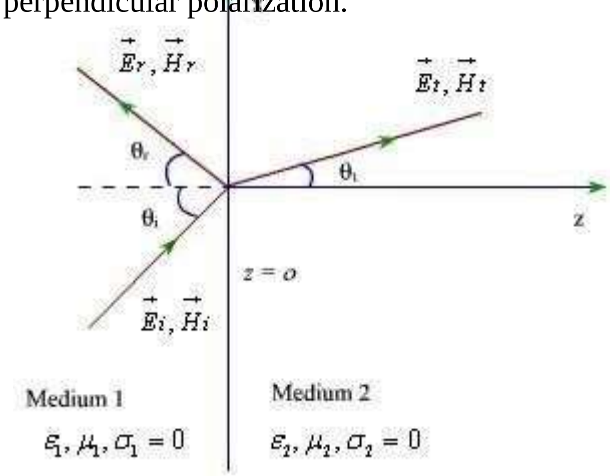

Fig 1.5: Oblique incidence at a plane dielectric interface
(www.brainkart.com/subject/Electromagnetic-Theory_206/)

For the case of a plane dielectric interface, an incident wave will be reflected partially and transmitted partially.

In Fig(1.5), θ_i, θ_0 and θ_t corresponds respectively to the angle of incidence, reflection and transmission.

1. **Parallel Polarization**

As discussed previously, the incident and reflected field components can be written as

$$\vec{E}_i(x,z) = E_{io}\left[\cos\theta_i \hat{a}_x - \sin\theta_i \hat{a}_z\right] e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

$$\vec{H}_i(x,z) = \hat{a}_y \frac{E_{io}}{\eta_1} e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

.... (5.82)

$$\vec{E}_r(x,z) = E_{ro}\left[\hat{a}_x \cos\theta_r + \hat{a}_z \sin\theta_r\right] e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

$$\vec{H}_r(x,z) = -\hat{a}_y \frac{E_{ro}}{\eta_1} e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

..........................(5.83)

In terms of the reflection coefficient Γ

$$\vec{E}_r(x,z) = \Gamma E_{io}\left[\hat{a}_x \cos\theta_r + \hat{a}_z \sin\theta_r\right] e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$
$$\vec{H}_r(x,z) = -\hat{a}_y \frac{\Gamma E_{io}}{\eta_1} e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$
. ...(5.84)

The transmitted filed can be written in terms of the transmission coefficient *T*

$$\vec{E}_t(x,z) = TE_{io}\left[\hat{a}_x \cos\theta_t - \hat{a}_z \sin\theta_t\right] e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)}$$
$$\vec{H}_t(x,z) = \hat{a}_y \frac{TE_{io}}{\eta_2} e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)}$$
..........................(5.85)

We can now enforce the continuity of tangential field components at the boundary i.e. *z=0*

$$\cos\theta_i e^{-j\beta_1 x\sin\theta_i} + \Gamma\cos\theta_r e^{-j\beta_1 x\sin\theta_r} = T\cos\theta_t e^{-j\beta_2 x\sin\theta_t}$$
$$\text{and } \frac{1}{\eta_1} e^{-j\beta_1 x\sin\theta_i} - \frac{\Gamma}{\eta_1} e^{-j\beta_1 x\sin\theta_r} = \frac{T}{\eta_2} e^{-j\beta_2 x\sin\theta_t}$$
..........................(5.85)

If both E_x and H_y are to be continuous at *z=0* for all *x* , then form the phase matching we have

$$\beta_1 \sin\theta_i = \beta_1 \sin\theta_r = \beta_2 \sin\theta_t$$

$\therefore$ We find that

$$\theta_i = \theta_r$$
$$\text{and } \beta_1 \sin\theta_i = \beta_2 \sin\theta_t$$
..........................(5.87)

Further, from equations (5.85) and (5.87) we have

$$\cos\theta_i + \Gamma\cos\theta_i = T\cos\theta_t$$
$$\text{and } \frac{1}{\eta_1} - \frac{\Gamma}{\eta_1} = \frac{T}{\eta_2}$$
..........................(5.88)

$$\therefore \quad \cos\theta_i(1+\Gamma) = T\cos\theta_t$$
$$\text{and } \frac{1}{\eta_1}(1-\Gamma) = \frac{T}{\eta_2}$$

$$\therefore T = \frac{\eta_2}{\eta_1}(1-\Gamma)$$

$$\cos\theta_i(1+\Gamma) = \frac{n_2}{n_1}(1-\Gamma)\cos\theta_t$$

$$\therefore (n_1\cos\theta_i + n_2\cos\theta_t)\Gamma = n_2\cos\theta_t - n_1\cos\theta_i$$

or $$\Gamma = \frac{n_2\cos\theta_t - n_1\cos\theta_i}{n_2\cos\theta_t + n_1\cos\theta_i}$$ ....(5.89)

and $$T = \frac{n_2}{n_1}(1-\Gamma)$$

$$= \frac{2n_2\cos\theta_i}{n_2\cos\theta_t + n_1\cos\theta_i}$$ ....(5.90)

From equation (5.90) we find that there exists specific angle $\theta_i = \theta_b$ for which Γ = *0* such that

$$n_2\cos\theta_t = n_1\cos\theta_b$$

or $$\sqrt{1-\sin^2\theta_t} = \frac{n_1}{n_2}\sqrt{1-\sin^2\theta_b}$$ ....(5.91)

Further, $$\sin\theta_t = \frac{\beta_1}{\beta_2}\sin\theta_b$$

$\mu_1 = \mu_2 = \mu_0$ ....(5.92)

For non magnetic material
Using this condition

$$1-\sin^2\theta_t = \frac{\varepsilon_1}{\varepsilon_2}(1-\sin^2\theta_b)$$

and $$\sin^2\theta_t = \frac{\varepsilon_1}{\varepsilon_2}\sin^2\theta_b$$ ....(5.93)

From equation (5.93), solving for $\sin\theta_b$ we get

$$\sin\theta_b = \frac{1}{\sqrt{1+\frac{\varepsilon_1}{\varepsilon_2}}}$$

This angle of incidence for which Γ = *0* is called Brewster angle. Since we are dealing with parallel polarization we represent this angle by $\theta_{b\|}$ so that

$$\sin\theta_{b\|} = \frac{1}{\sqrt{1+\frac{\varepsilon_1}{\varepsilon_2}}}$$

2. Perpendicular Polarization

For this case

$$\vec{E}_i(x,z) = \hat{a}_y E_{io} e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

$$\vec{H}_i(x,z) = \frac{E_{io}}{\eta_1}\left[-\hat{a}_x\cos\theta_i + \hat{a}_z\sin\theta_i\right]e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)} \quad \text{.........................(5.94)}$$

$$\vec{E}_r(x,z) = \hat{a}_y \Gamma E_{io} e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

$$\vec{H}_r(x,z) = \frac{\Gamma E_{io}}{\eta_1}\left[\hat{a}_x\cos\theta_r + \hat{a}_z\sin\theta_r\right]e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)} \quad \text{.........................(5.95)}$$

$$\vec{E}_t(x,z) = \hat{a}_y T E_{io} e^{-j\beta_1(x\sin\theta_t + z\cos\theta_t)}$$

$$\vec{H}_t(x,z) = \frac{T E_{io}}{\eta_2}\left[-\hat{a}_x\cos\theta_t + \hat{a}_z\sin\theta_t\right]e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)} \quad \text{.........................(5.95)}$$

Using continuity of field components at *z=0*

$$e^{-j\beta_1 x\sin\theta_i} + \Gamma e^{-j\beta_1 x\sin\theta_r} = T E_{io} e^{-j\beta_2 x\sin\theta_t}$$

$$\text{and } -\frac{1}{\eta_1}\cos\theta_i e^{-j\beta_1 x\sin\theta_i} + \frac{\Gamma}{\eta_1}\cos\theta_r e^{-j\beta_1 x\sin\theta_r} = -\frac{T}{\eta_2}\cos\theta_t e^{-j\beta_2 x\sin\theta_t} \quad \text{.........................(5.97)}$$

As in the previous case

$$\beta_1\sin\theta_i = \beta_1\sin\theta_r = \beta_2\sin\theta_t$$

$$\therefore \quad \theta_i = \theta_r$$

$$\text{and } \sin\theta_t = \frac{\beta_1}{\beta_2}\sin\theta_i \quad \text{.........................(5.98)}$$

Using these conditions we can write

$$1 + \Gamma = T$$

$$-\frac{\cos\theta_i}{\eta_1} + \frac{\Gamma\cos\theta_i}{\eta_1} = -\frac{T\cos\theta_t}{\eta_2} \quad \text{.........................(5.99)}$$

From equation (5.99) the reflection and transmission coefficients for the perpendicular polarization can be computed as

$$\Gamma = \frac{n_2 \cos\theta_i - n_1 \cos\theta_t}{n_2 \cos\theta_i + n_1 \cos\theta_t}$$

$$\text{and} \quad T = \frac{2n_2 \cos\theta_i}{n_2 \cos\theta_i + n_1 \cos\theta_t} \quad \text{.........................(5.100)}$$

We observe that if Γ = *0* for an angle of incidence $\theta_i = \theta_b$

$$n_2 \cos\theta_b = n_1 \cos\theta_t$$

$$\text{or} \quad \sin^2\theta_b \left(\frac{\mu_1\varepsilon_1}{\mu_2\varepsilon_2} - \frac{\mu_2\varepsilon_1}{\mu_1\varepsilon_2} \right) = \left(1 - \frac{\mu_2\varepsilon_1}{\mu_1\varepsilon_2} \right)$$

$$\therefore \cos^2\theta_t = \frac{n_2}{n_1} \cos^2\theta_b$$

$$= \frac{\mu_2\varepsilon_1}{\mu_1\varepsilon_2} \cos^2\theta_b$$

$$\therefore 1 - \sin^2\theta_t = \frac{\mu_2\varepsilon_1}{\mu_1\varepsilon_2}\left(1 - \sin^2\theta_b\right)$$

$$\text{Again} \quad \sin\theta_t = \frac{\beta_1}{\beta_2} \sin\theta_b$$

$$\therefore \sin^2\theta_t = \frac{\mu_1\varepsilon_1}{\mu_2\varepsilon_2} \sin^2\theta_b$$

$$\therefore \left(1 - \frac{\mu_1\varepsilon_1}{\mu_2\varepsilon_2} \sin^2\theta_b \right) = \frac{\mu_2\varepsilon_1}{\mu_1\varepsilon_2} - \frac{\mu_2\varepsilon_1}{\mu_1\varepsilon_2} \sin^2\theta_b$$

$$\text{or} \quad \sin^2\theta_b \left(\frac{\mu_1^2 - \mu_2^2}{\mu_1\mu_2\varepsilon_2} \right) \varepsilon_1 = \left(\frac{\mu_1\varepsilon_2 - \mu_2\varepsilon_1}{\mu_1\varepsilon_2} \right)$$

$$\text{or.} \sin^2\theta_b = \frac{\mu_2\left(\mu_1\varepsilon_2 - \mu_2\varepsilon_1\right)}{\varepsilon_1\left(\mu_1^2 - \mu_2^2\right)} \quad \text{.......................(5.101)}$$

We observe if $\mu_1 = \mu_2 = \mu_0$ i.e. in this case of non magnetic material Brewster angle does not exist as the denominator or equation (5.101) becomes zero. Thus for perpendicular polarization in dielectric media, there is Brewster angle so that can be made equal to zero.

From our previous discussion we observe that for bothpolarizations

$$\sin\theta_t = \frac{\beta_1}{\beta_2}\sin\theta_i$$

$\mu_1 = \mu_2 = \mu_0$

$$\sin\theta_t = \sqrt{\frac{\varepsilon_1}{\varepsilon_2}}\sin\theta_i$$

For $\varepsilon_1 > \varepsilon_2$, $\theta_t > \theta_i$

The incidence angle $\theta_i = \theta_r$ for which $\theta_t = \frac{\pi}{2}$ i.e. $\theta_c = \sin^{-1}\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}$ is called the critical angle ofincidence.

Plane waves in a lossy medium

In a lossy medium, the EM wave looses power as it propagates. Such a mediumis conducting with conductivity σ and we can write:

$$\nabla \times \vec{H} = \vec{J} + j\omega\varepsilon\vec{E} = (\sigma + j\omega\varepsilon)\vec{E}$$
$$= j\omega\left(\varepsilon + \frac{\sigma}{j\omega}\right)\vec{E}$$
$$= j\omega\varepsilon_c\vec{E} \qquad \text{.....................(5.19)}$$

Where $\varepsilon_c = \varepsilon - j\dfrac{\sigma}{\omega} = \varepsilon' - j\varepsilon''$ is called the complex permittivity.

We have already discussed how an external electric field can polarize a dielectric and give rise to bound charges. When the external electric field is time varying, the polarization vector will vary with the same frequency as that of the applied field. As the frequency of the applied filed increases, the inertia of the charge particles tend to prevent the particle displacement keeping pace with the applied field changes. This results in frictional damping mechanism causing power loss.

In addition, if the material has an appreciable amount of free charges, there will be ohmic losses. It is customary to include the effect of damping and ohmic losses in the imaginary part of ε_c. An equivalent conductivity $\sigma = \omega\varepsilon''$ represents all losses.

The ratio $\dfrac{\varepsilon''}{\varepsilon'}$ is called loss tangent as this quantity is a measure of the power loss.

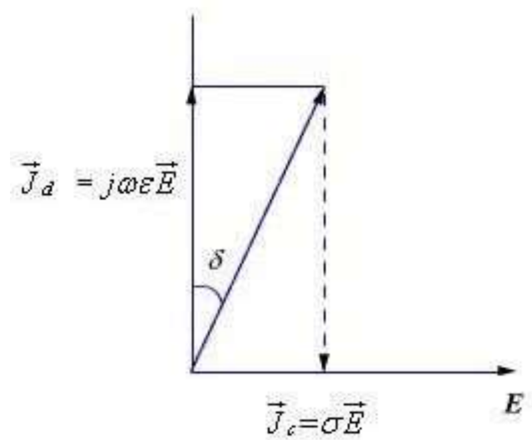

Fig 3.1 : Calculation of Loss Tangent

(www.brainkart.com/subject/Electromagnetic-Theory_206/)

With reference to the Fig 3.1,

$$\tan\delta = \frac{|\vec{J}_c|}{|\vec{J}_d|} = \frac{\sigma}{\omega\varepsilon} = \frac{\varepsilon''}{\varepsilon'} \quad \text{.......................... (5.20)}$$

where $\vec{J}_c$ is the conduction current density and $\vec{J}_d$ is displacement current density. The loss tangent gives a measure of how much lossy is the medium under consideration. For a good dielectric medium $(\sigma << \omega\varepsilon)$, $\tan\delta$ is very small and the medium is a good conductor if $(\sigma \gg \omega\varepsilon)$. A material may be a good conductor at low frequencies but behave as lossy dielectric at higher frequencies.

For a source free lossy medium we can write

$$\left.\begin{array}{ll} \nabla\times\vec{H} = (\sigma + j\omega\varepsilon)\vec{E} & \nabla.\vec{H} = 0 \\ \nabla\times\vec{E} = -j\omega\mu\vec{H} & \nabla.\vec{E} = 0 \end{array}\right\} \quad \text{..........................(5.21)}$$

$$\nabla\times\nabla\times\vec{E} = \nabla(\nabla.\vec{E}) - \nabla^2\vec{E} = -j\omega\mu\nabla\times\vec{H} = -j\omega\mu(\sigma + j\omega\varepsilon)\vec{E}$$

$$or, \nabla^2\vec{E} - \gamma^2\vec{E} = 0 \quad \text{.................... (5.22)}$$

Where $\gamma^2 = j\omega\mu(\sigma + j\omega\varepsilon)$

Proceeding in the same manner we can write,

$$\nabla^2\vec{H} - \gamma^2\vec{H} = 0$$

$$\gamma = \alpha + i\beta = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)} = j\omega\sqrt{\mu\varepsilon}\left(1 + \frac{\sigma}{j\omega\varepsilon}\right)^{1/2}$$

is called the propagation constant.

The real and imaginary parts α and β of the propagation constant η can be computed as follows:

$$\gamma^2 = (\alpha + i\beta)^2 = j\omega\mu(\sigma + j\omega\varepsilon)$$

$$or, \alpha^2 - \beta^2 = -\omega^2\mu\varepsilon$$

And $$\alpha\beta = \frac{\omega\mu\sigma}{2}$$

$$\therefore \alpha^2 - \left(\frac{\omega\mu\sigma}{2\alpha}\right)^2 = -\omega^2\mu\varepsilon$$

$$or, 4\alpha^4 + 4\alpha^2\omega^2\mu\varepsilon = \omega^2\mu^2\sigma^2$$

$$or, 4\alpha^4 + 4\alpha^2\omega^2\mu\varepsilon + \omega^4\mu^2\varepsilon^2 = \omega^2\mu^2\sigma^2 + \omega^4\mu^2\varepsilon^2$$

$$or, \left(2\alpha^2 + \omega^2\mu\varepsilon\right)^2 = \omega^4\mu^2\varepsilon^2\left(1 + \frac{\sigma^2}{\omega^2\varepsilon^2}\right)$$

$$or, \alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} - 1\right]} \quad \text{.................. (5.23a)}$$

Similarly $\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} + 1\right]}$ ....(5.23b)

Let us now consider a plane wave that has only x -component of electric field and propagate along z .

$$\therefore \vec{E}_x(z) = \left(E_0^+ e^{-\gamma z} + E_0^- e^{\gamma z}\right)\hat{a}_x \quad \text{.................................. (5.24)}$$

Considering only the forward traveling wave

$$\vec{\varepsilon}(z,t) = \operatorname{Re}\left(E_0^+ e^{-\gamma z} e^{j\omega t}\right)\hat{a}_x$$

$$= E_0^+ e^{-\alpha z}\cos\left(\omega t - \beta z\right)\hat{a}_x \quad \text{.................................. (5.25)}$$

Similarly, from $\vec{\mathcal{H}} = -\frac{1}{j\omega\mu}\nabla \times \vec{E}$, we can find

$$\vec{\mathcal{H}}(z,t) = \frac{E_0}{\eta} e^{-\alpha z}\cos\left(\omega t - \beta z\right)\hat{a}_y \quad \text{.................................... (5.25)}$$

Where $\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} = |\eta| e^{j\theta_\eta}$

$$\therefore \vec{\mathcal{H}} = \frac{E_0}{|\eta|} e^{-\alpha z}\cos\left(\omega t - \beta z - \theta_\eta\right)\hat{a}_y \quad \text{.................................... (5.27)}$$

From (5.25) and (5.25) we find that as the wave propagates along z, it decreases in amplitude by a factor $e^{-\alpha z}$. Therefore α is known as attenuation constant. Further $\vec{E}$ and $\vec{H}$

are out of phase by an angle θ_n.

For low loss dielectric, $\frac{\sigma}{\omega\varepsilon} << 1$, i.e., $\varepsilon'' << \varepsilon'$.

Using the above condition approximate expression for α and β can be obtained as follows:

$$\gamma = \alpha + i\beta = j\omega\sqrt{\mu\varepsilon'}\left[1 - j\frac{\varepsilon''}{\varepsilon'}\right]^{1/2}$$

$$\cong j\omega\sqrt{\mu\varepsilon'}\left[1 - j\frac{1}{2}\frac{\varepsilon''}{\varepsilon'} + \frac{1}{8}\left(\frac{\varepsilon''}{\varepsilon'}\right)^2\right]$$

$$\left.\begin{aligned} \alpha &= \frac{\omega\varepsilon''}{2}\sqrt{\frac{\mu}{\varepsilon'}} \\ \beta &= \omega\sqrt{\mu\varepsilon'}\left[1 + \frac{1}{8}\left(\frac{\varepsilon''}{\varepsilon'}\right)^2\right] \end{aligned}\right\} \text{........(6.28)}$$

$$\eta = \sqrt{\frac{\mu}{\varepsilon'}}\left(1 - j\frac{\varepsilon''}{\varepsilon'}\right)^{-1/2}$$

$$= \sqrt{\frac{\mu}{\varepsilon'}}\left(1 + j\frac{\varepsilon''}{2\varepsilon'}\right) \text{ .... (5.29)}$$

& phase velocity

$$v_p = \frac{\omega}{\beta} \cong \frac{1}{\sqrt{\mu\varepsilon'}}\left[1 - \frac{1}{8}\left(\frac{\varepsilon''}{\varepsilon'}\right)^2\right] \text{ .... (5.30)}$$

For good conductors $\frac{\sigma}{\omega\varepsilon} >> 1$

$$\gamma = j\omega\sqrt{\mu\varepsilon}\left(1 + \frac{\sigma}{j\omega\varepsilon}\right) \cong j\omega\sqrt{\mu\varepsilon}\sqrt{\frac{\sigma}{j\omega\varepsilon}}$$

$$= \frac{1+j}{\sqrt{2}}\sqrt{\omega\mu\sigma} \text{ ....(5.31)}$$

We have used the relation

$$\sqrt{j} = \left(e^{j\pi/2}\right)^{1/2} = e^{j\pi/4} = \frac{1}{\sqrt{2}}(1+j)$$

From (5.31) we can write

$$\alpha + i\beta = \sqrt{\pi f \mu \sigma} + j\sqrt{\pi f \mu \sigma}$$

$$\therefore \alpha = \beta = \sqrt{\pi f \mu \sigma} \quad \text{............... (5.32)}$$

$$\eta = \sqrt{\frac{j\omega\mu}{j\omega\varepsilon\left(1+\frac{\sigma}{j\omega\varepsilon}\right)}}$$

$$\cong \sqrt{\frac{\mu}{\varepsilon}\frac{j\omega\varepsilon}{\sigma}} = \sqrt{\frac{j\omega\mu}{\sigma}}$$

$$= (1+j)\sqrt{\frac{\pi f \mu}{\sigma}}$$

$$v_p = \frac{\omega}{\beta} \cong \sqrt{\frac{2\omega}{\mu\sigma}} \quad \text{..................... (5.33)}$$

TEM Waves

Let us now consider the propagation of a uniform plane wave in any arbitrary direction that doesn't necessarily coincides with an axis.

For a uniform plane wave propagating in z-direction

$\vec{E}(z) = E_0 e^{-jkz}$, E_0 is a constant vector ....(6.11)

The more general form of the above equation is

$$\vec{E}(x,y,z) = \vec{E}_0 e^{-jk_x x - jk_y y - jk_z z}$$ ....(6.12

$$\nabla^2 \vec{E} + k^2 \vec{E} = 0$$

This equation satisfies Helmholtz's equationprovided,

$$k_x^2 + k_y^2 + k_z^2 = k^2 = \omega^2 \mu \varepsilon$$ .... (6.13)

We define wave number vector $\vec{k} = \hat{a}_x k_x + \hat{a}_y k_y + \hat{a}_z k_z = k \hat{a}_n$ ....(6.14)

And radius vector from the origin

$$\vec{r} = \hat{a}_x x + \hat{a}_y y + \hat{a}_z z$$ .... (6.15)

Therefore we can write

$$\vec{E}(\vec{r}) = \vec{E}_0 e^{-j\vec{k}\cdot\vec{r}} = \vec{E}_0 e^{-jk\hat{a}_n \cdot \vec{r}}$$ .... (6.16)

Here $\hat{a}_n \cdot \vec{r}$ =constant is a plane of constant phase and uniform amplitude just in the case of $\vec{E}(z) = \vec{E}_0 e^{-jkz}$,

z =constant denotes a plane of constant phase and uniform

amplitude. If the region under consideration is charge free,

$$\nabla \cdot \vec{E} = 0$$

$$\therefore \nabla \cdot \left(\vec{E}_0 e^{-j\vec{k}\cdot\vec{r}}\right) = 0$$

Using the vector identity $\nabla \cdot \left(f\vec{A}\right) = \vec{A}\nabla f + f\nabla \cdot \vec{A}$ and noting that $\vec{E}_0$ is constant we can write,

$$\vec{E}_0.\nabla\left(e^{-jk\hat{a}_n.\vec{r}}\right) = 0$$

$$or,\ \vec{E}_0.\left[\left(\frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z\right)e^{-j(k_x x + k_y y + k_z z)}\right] = 0$$

$$or,\ \vec{E}_0.\left(-jk\,\hat{a}_n e^{-jk\hat{a}_n.\vec{r}}\right) = 0$$

$$\vec{E}_0.\hat{a}_n = 0 \qquad \text{......................(6.17)}$$

i.e., $\vec{E}_0$ is transverse to the direction of thepropagation.

The corresponding magnetic field can be computed asfollows:

$$\vec{H}\left(\vec{r}\right) = -\frac{1}{j\omega\mu}\nabla\times\vec{E}\left(\vec{r}\right) = -\frac{1}{j\omega\mu}\nabla\times\left(\vec{E}_0 e^{-j\vec{k}.\vec{r}}\right)$$

$$\nabla\times\left(\psi\vec{A}\right) = \psi\nabla\times\vec{A} + \nabla\psi\times\vec{A}$$

$$\vec{H}(\vec{r}) = -\frac{1}{j\omega\mu}\nabla e^{-j\vec{k}.\vec{r}}\times\vec{E}_0$$

$$= -\frac{1}{j\omega\mu}\left[-jk\,\hat{a}_n\times\vec{E}_0 e^{-jk\hat{a}_n.\vec{r}}\right]$$

$$= \frac{k}{\omega\mu}\hat{a}_n\times\vec{E}\left(\vec{r}\right)$$

$$\vec{H}(\vec{r}) = \frac{1}{\eta}\hat{a}_n\times\vec{E}\left(\vec{r}\right) \qquad \text{.....................(6.18)}$$

η is the intrinsic impedance of the medium. We observe $\vec{H}(\vec{r})$ is perpendicular

to both $\hat{a}_n$ and $\vec{E}\left(\vec{r}\right)$. Thus the electromagnetic wave represented by $\vec{E}\left(\vec{r}\right)$ and $\vec{H}(\vec{r})$ is aTEM wave.

Wave equation and their solution

From equation 4.24 we can write the Maxwell's equations in the differential form as

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$
$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$
$$\nabla \cdot \vec{D} = \vec{\rho}$$
$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{H} = \sigma \vec{E} + \varepsilon \frac{\partial \vec{E}}{\partial t} \qquad (4.29(a))$$
$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \qquad (4.29(b))$$
$$\nabla \cdot \vec{E} = 0 \qquad (4.29(c))$$
$$\nabla \cdot \vec{H} = 0 \qquad (4.29(d))$$

Let us consider a source free uniform medium having dielectric constant , magnetic permeability μ and conductivity . The above set of equations can be written as

Using the vector identity ,

$$\nabla \times \nabla \times \vec{A} = \nabla \cdot \left(\nabla \cdot \vec{A}\right) - \nabla^2 A$$

We can write from 4.29(b)

$$\nabla \times \nabla \times \vec{E} = \nabla \cdot \left(\nabla \cdot \vec{E}\right) - \nabla^2 \vec{E}$$
$$= -\nabla \times \left(\mu \frac{\partial \vec{H}}{\partial t}\right)$$

or $$\nabla \times \vec{H} \nabla \cdot \left(\nabla \cdot \vec{E}\right) - \nabla^2 \vec{E} = -\mu \frac{\partial}{\partial t}\left(\nabla \times \vec{H}\right)$$

Substituting from 4.29(a)

$$\nabla\cdot\left(\nabla\cdot\vec{E}\right)-\nabla^2\vec{E}=-\mu\frac{\partial}{\partial t}\left(\sigma\vec{E}+\varepsilon\frac{\partial\vec{E}}{\partial t}\right)$$

But in source free medium $\nabla\cdot\vec{E}=0$ (eqn 4.29(c))

$$\nabla^2\vec{E}=\mu\sigma\frac{\partial\vec{E}}{\partial t}+\mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2}$$ (4.30)

In the same manner for equation eqn 4.29(a)

$$\nabla\times\nabla\times\vec{H}=\nabla\cdot\left(\nabla\cdot\vec{H}\right)-\nabla^2\vec{H}$$

$$=\sigma\left(\nabla\times\vec{E}\right)+\varepsilon\frac{\partial}{\partial t}\left(\nabla\times\vec{E}\right)$$

$$=\sigma\left(-\mu\frac{\partial\vec{H}}{\partial t}\right)+\varepsilon\frac{\partial}{\partial t}\left(-\mu\frac{\partial\vec{H}}{\partial t}\right)$$

Since $\nabla\cdot\vec{H}=0$ from eqn 4.29(d), we can write

$$\nabla^2\vec{H}=\mu\sigma\left(\frac{\partial\vec{H}}{\partial t}\right)+\mu\varepsilon\left(\frac{\partial^2\vec{H}}{\partial t^2}\right) \quad (4.31)$$

These two equations

$$\nabla^2\vec{E}=\mu\sigma\frac{\partial\vec{E}}{\partial t}+\mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2}$$

$$\nabla^2\vec{H}=\mu\sigma\left(\frac{\partial\vec{H}}{\partial t}\right)+\mu\varepsilon\left(\frac{\partial^2\vec{H}}{\partial t^2}\right)$$

are known as wave equations.

It may be noted that the field components are functions of both space and time. Forexample, if we consider a Cartesian co ordinate system, $\vec{E}$ and $\vec{H}$ essentially represents $\vec{E}(x,y,z,t)$ and $\vec{H}(x,y,z,t)$. For simplicity, we consider propagation in free space $\sigma=0$, $\mu=\mu_0$, $\varepsilon=\varepsilon_0$

The wave eqn in equations 4.30 and 4.31 reduces to

$$\nabla^2 \vec{E} = \mu_0 \varepsilon_0 \left(\frac{\partial^2 \vec{E}}{\partial t^2} \right) \qquad (4.32(a))$$

$$\nabla^2 \vec{H} = \mu_0 \varepsilon_0 \left(\frac{\partial^2 \vec{H}}{\partial t^2} \right) \qquad (4.32(b))$$

Further simplifications can be made if we consider in Cartesian co ordinate system aspecial case where $\vec{E}$ and $\vec{H}$ are considered to be independent in two dimensions, say

$\vec{E}$ and $\vec{H}$ are assumed to be independent of *y* and *z*. Such waves are called plane waves.

From eqn (4.32 (a)) we can write

$$\frac{\partial^2 \vec{E}}{\partial x^2} = \varepsilon_0 \mu_0 \left(\frac{\partial^2 \vec{E}}{\partial t^2} \right)$$

The vector wave equation is equivalent to the three scalarequations

$$\frac{\partial^2 \vec{E_x}}{\partial x^2} = \varepsilon_0 \mu_0 \left(\frac{\partial^2 \vec{E_x}}{\partial t^2} \right) \qquad (4.33(a))$$

$$\frac{\partial^2 \vec{E_y}}{\partial x^2} = \varepsilon_0 \mu_0 \left(\frac{\partial^2 \vec{E_y}}{\partial t^2} \right) \qquad (4.33(b))$$

$$\frac{\partial^2 \vec{E_z}}{\partial x^2} = \varepsilon_0 \mu_0 \left(\frac{\partial^2 \vec{E_z}}{\partial t^2} \right) \qquad (4.33(c))$$

Since we have ,

$$\nabla \cdot \vec{E} = 0 \quad \therefore \frac{\partial \vec{E_x}}{\partial x} + \frac{\partial \vec{E_y}}{\partial y} + \frac{\partial \vec{E_z}}{\partial z} = 0 \qquad (4.34)$$

As we have assumed that the field components are independent of y and z eqn(4.34)reduces to $\frac{\partial E_x}{\partial x} = 0$

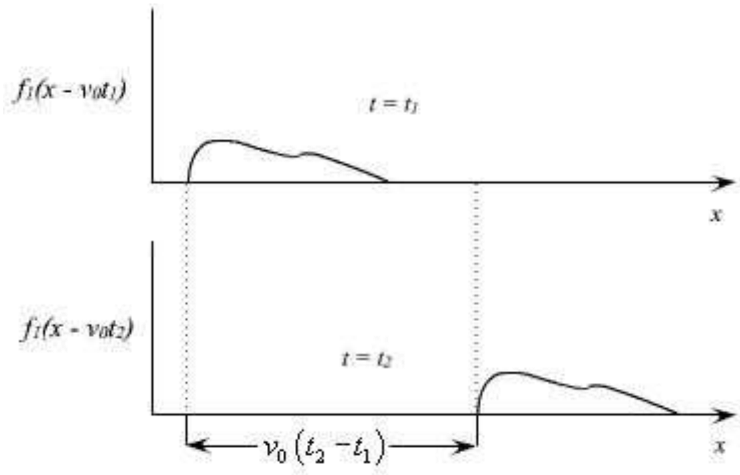

Fig 4.1 : Traveling wave in the + x direction

(www.brainkart.com/subject/Electromagnetic-Theory_206/)

A field component satisfying either of the last two conditions (i.e (ii) and (iii))is not a part of a plane wave motion and hence E_X is taken to be equal to zero. Therefore, a uniform plane wave propagating in x direction does not have a field component (*E* or *H*) acting along x as shown in figure 4.1.

Without loss of generality let us now consider a plane wave having E_y component only(Identical results can be obtained for E_Z component) .

The equation involving such wave propagation is given by

$$\frac{\partial^2 \overrightarrow{E_y}}{\partial x^2} = \varepsilon_0 \mu_0 \left(\frac{\partial^2 \overrightarrow{E_y}}{\partial t^2} \right) \qquad (4.36)$$

The above equation has a solution of the form

$$E_y = f_1(x - v_0 t) + f_2(x + v_0 t) \qquad (4.37)$$

where $v_0 = \dfrac{1}{\sqrt{\mu_0 \varepsilon_0}}$

Thus equation (4.37) satisfies wave eqn (4.36) can be verified by substitution.

$f_1(x - v_0 t)$ corresponds to the wave traveling in the + x direction while $f_2(x + v_0 t)$

corresponds to a wave traveling in the -x direction. The general solution of the wave eqn thus consists of two waves, one traveling away from the source and other traveling back towards the source. In the absence of any reflection, the second form of the eqn (4.37) is zero and the solution can be written as

$$E_y = f_1(x - v_0 t) \quad (4.38)$$

Such a wave motion is graphically shown in fig 4.4 at two instances of time t1 and t2.

Let us now consider the relationship between E and H components for the forwardtraveling wave.

Since $\vec{E} = \hat{a}_y E_y = \hat{a}_y f_1(x - v_0 t)$ and there is no variation along y and z.

$$\nabla \times \vec{E} = \hat{a}_z \frac{\partial E_y}{\partial x}$$

Since only z component of $\nabla \times \vec{E}$ exists, from (4.29(b))

$$\frac{\partial E_y}{\partial x} = -\mu_0 \frac{\partial H_z}{\partial t} \quad (4.39)$$

and from (4.29(a)) with $\sigma = 0$, only H_z component of magnetic field being present

$$\nabla \times \vec{H} = -\hat{a}_y \frac{\partial H_z}{\partial x}$$

$$\therefore -\frac{\partial H_z}{\partial x} = \varepsilon_0 \frac{\partial E_y}{\partial t}$$

Substituting Eq (4.38)

$$\therefore H_z = \sqrt{\frac{\varepsilon_0}{\mu_0}} \cdot \int f_1'(x - v_0 t) dx + c$$

$$\frac{\partial H_z}{\partial x} = -\varepsilon_0 \frac{\partial E_y}{\partial t} = \varepsilon_0 v_0 f_1'\left(x - v_0 t\right)$$

$$\therefore \frac{\partial H_z}{\partial x} = \varepsilon_0 \frac{1}{\sqrt{\mu_0 \varepsilon_0}} f_1'\left(x - v_0 t\right)$$

$$= \sqrt{\frac{\varepsilon_0}{\mu_0}} \int \frac{\partial}{\partial x} f_1 dx + c$$

$$= \sqrt{\frac{\varepsilon_0}{\mu_0}} f_1 + c \qquad (4.40)$$

The constant of integration means that a field independent of x may also exist. However,this field will not be a part of the wave motion.

Hence

$$H_z = \sqrt{\frac{\varepsilon_0}{\mu_0}} E_y \qquad (4.41)$$

$$H_z = \sqrt{\frac{\varepsilon_0}{\mu_0}} E_y + c$$

which relates the *E* and *H* components of the traveling wave.

$$z_0 = \frac{E_y}{H_z} = \sqrt{\frac{\mu_0}{\varepsilon_0}} \cong 120\pi \text{ or } 377\Omega$$

$z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}$ is called the characteristic or intrinsic impedance of the free space

Harmonic fields

In the previous section we introduced the equations pertaining to wave propagation and discussed how the wave equations are modified for time harmonic case. In this section we discuss in detail a particular form of electromagnetic wave propagation called'plane waves'.

The Helmhotz Equation:

In source free linear isotropic medium, Maxwell equations in phasor form are,

$$\nabla \times \vec{E} = -j\omega\mu\vec{H} \qquad \nabla \times \vec{E} = 0$$

$$\nabla \times \vec{H} = j\omega\varepsilon\vec{E} \qquad \nabla \times \vec{H} = 0$$

$$-\nabla^2\vec{E} = -j\omega\mu(j\omega\varepsilon\vec{E})$$

or,

$$\therefore \nabla \times \nabla \times \vec{E} = \nabla\left(\nabla \times \vec{E}\right) - \nabla^2\vec{E} = -j\omega\mu\nabla \times \vec{H}$$

or,

$$\nabla^2\vec{E} + \omega^2\mu\varepsilon\vec{E} = 0$$

or, $\nabla^2 \vec{E} + k^2 \vec{E} = 0$ where $k = \omega\sqrt{k\varepsilon}$

An identical equation can be derived for $\vec{H}$.

i.e., $\nabla^2 \vec{H} + k^2 \vec{H} = 0$

These equations

$$\left.\begin{aligned} \nabla^2 \vec{E} + k^2 \vec{E} = 0 \ldots\ldots\ldots\ldots(a) \\ \& \quad \nabla^2 \vec{H} + k^2 \vec{H} = 0 \ldots\ldots\ldots\ldots(b) \end{aligned}\right\} \ldots\ldots \quad (4.42)$$

are called homogeneous vector Helmholtz's equation.

$k = \omega\sqrt{\mu\varepsilon}$ is called the wave number or propagation constant of the medium.